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Robinson's Shorter Course.

A COLLECTION

OF

ALGEBRAIC PROBLEMS,

DESIGNED

FOR DRILL AND REVIEW IN CONNECTION WITH ANY SYSTEMATIC TREATISE ON ALGEBRA.

FOR SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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IVISON, BLAKEMAN, TAYLOR & CO.,

NEW YORK AND CHICAGO.

1875.

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PREFACE.

TEACHERS of Algebra have long felt the want of a large and well-arranged collection of algebraic problems. The problems of the text-book used may not be sufficiently numerous or of sufficient variety, or the teacher may wish to test the student's knowledge of principles by requiring him to solve some problem for which he has made no special preparation; or after the problems of the text-book have been used for many years, and solutions have been handed down from one class to another, a new set may be desirable.

To meet these wants and to diminish the labor of the teacher, is the object of the present collection.

This collection of problems has been made to conform, in arrangement and classification, to the author's Complete Algebra, but it may be used in connection with any other algebra.

An Appendix containing the treatment of Continued Fractions, Reciprocal Equations, Elimination by the Method of the Greatest Common Divisor, Cardan's Formula for Cubic Equations, and other matter, has been added.

The materials for this volume have been drawn, in part, from the works of Barnard Smith, Bland, Wrigley, Goodwin, Todhunter, the Schoolday Magazine, and the Analyst.

Solutions of all the difficult problems of this collection will be found in the "Key to Ficklin's Complete Algebra."

THE AUTHOR.

UNIVERSITY OF THE STATE OF MISSOURI,
Columbia, April, 1875.

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ALGEBRAIC PROBLEMS.

NOTATION.

1. Express, in algebraic language, the following eight statements·

1. The square of a increased by four times b.

Ans. $a^2 + 4b$.

2. Seven times the product of x and y, diminished by five times the cube of z. *Ans.* $7xy - 5z^3$.

3. Twelve times the square of a minus five times the cube of b, divided by the sum of a and c. *Ans.* $\frac{12a^2 - 5b^3}{a + c}$.

4. The product of a, c, and $a - c$. *Ans.* $ac(a - c)$.

5. One-third of x, increased by a, is equal to three times y diminished by b.

Ans. $\frac{x}{3} + a = 3y - b$.

6. The square root of the difference between the square of a and the cube of b is equal to x minus the cube root of the sum of x and a. *Ans.* $\sqrt{a^2 - b^3} = x - \sqrt[3]{x + a}$.

7. The square of a, increased by the square of b, is greater than twice the product of a and b.

Ans. $a^2 + b^2 > 2ab$.

8. The n^{th} power of x, increased by the n^{th} root of the binomial a plus x, is less than twice the square of m divided by the m^{th} root of the binomial a minus b.

Ans. $x^n + \sqrt[n]{a} + \bar{x} < \frac{2m^2}{\sqrt[m]{a-b}}$.

Express, in common language, the following eight algebraic expressions:

9. $(a^2 - bc)\,a$, $(a + bd)\,m$, $am + c^2 - md^4$, $(a+b)m - (c+d)\,n$, $[4a^2 - (3b^2 - 2c)]\,d$, $\frac{2b^2+c^2}{md^2} \times (a-c)$, $a + x - \{b + y - [a - x - \overline{b - 2y}]\}$, $\frac{\sqrt{(x+y)^3}}{\sqrt[3]{(a+x)^2}}$.

NUMERICAL VALUES.

2. If $a = 8$, $b = 6$, $c = 4$, $d = 2$, $m = 3$, $n = 1$, find the numerical value of each of the following expressions:

1. $\frac{a^2 + b^2 + c^2 + d^2}{a + b + c + d}$. *Ans.* 6.

2. $\frac{(6a^2n - 4m^3d)\,(m^2 - 7n)}{m}$. *Ans.* 112.

3. $\frac{(5an + 26)\,d^3}{a + b + c + d + m + n}$. *Ans.* 22.

4. $\left(\frac{5a}{c} - \frac{3c}{d^2}\right)a$. *Ans.* 56.

5. $\left[\frac{(a + b - c)\,(a - b + c)}{m^2 - c\,(n+1) + 1}\right](m - d)$. *Ans.* 30.

6. $\frac{am^2}{mc} + \frac{d^4 - 1}{m(m^2 - b)}$. *Ans.* $7\frac{2}{3}$.

7. $\frac{\overline{[a + 2c} \times m - d^2]\, m - 2(ab + m^2)}{c}$. *Ans.* $41\frac{1}{2}$.

8. $\left(\frac{a}{b} + \frac{m}{c} + \frac{d}{m}\right) 2b$. *Ans.* 33.

9. $\frac{abcd + a + b + c + d}{m + n} \times \frac{c}{d}$. *Ans.* 202.

10. $\frac{\sqrt{m + n}}{\sqrt[3]{b + d}} + \sqrt{\frac{m + n}{c}}$. *Ans.* 2.

ADDITION.

3.—1. Find the sum of $2a^2 - 3c^2$, $4a^2 - 7c^2$, $7a^2 - 4c^2$, $a^2 - 3c^2$, $3a^2 - c^2$. *Ans.* $17a^2 - 18c^2$.

2. Find the sum of $2ab - 3c^2 + 2b^2$, $ab - 7c^2 + b^2$, $7ab - c^2 + 7b^2$, $10ab - 6c^2 + 5b^2$, $5ab - 5c^2 + 6b^2$. *Ans.* $25ab - 22c^2 + 21b^2$.

3. Find the sum of $4x^3 - 8xy - 5y^2 + 7$, $8x^3 - 4xy - 7y^2 + 13$, $5x^3 - xy - y^2 + 1$, $21x^3 - 10xy - 21y^2 + 10$, $x^3 - 4xy - 5y^2 + 2$. *Ans.* $39x^3 - 27xy - 39y^2 + 33$.

4. Find the sum of $5x^3 - 3x + 2y$, $-x^3 + 2x - y$, $7x^3 - 4x + 3y$. *Ans.* $11x^3 - 5x + 4y$.

5. Find the sum of $2a^5 - 8a^4b - 14b^5$, $3a^5 + 7a^4b - 27b^5$, $-15a^5 + a^4b + 8b^5$, $32a^5 - 15a^4b + 32b^5$, $-17a^5 + 32a^4b - 17b^5$. *Ans.* $5a^5 + 17a^4b - 18b^5$.

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23. Divide $x^2 - (a - b)\,x - ab$ by $x + b$.

Ans. $x - a$.

24. Divide $a^2b - bx^2 + a^2x - x^3$ by $(x + b)\,(a - x)$.

Ans. $a + x$.

25. Divide $x^3 - (a + c)\,x^2 + (b + ac)\,x - bc$ by $x - c$.

Ans. $x^2 - ax + b$.

26. Write the respective quotients of $n^2 - 1$ by $n + 1$, $x^3 - y^3$ by $x - y$, $36x^2 - 16$ by $6x - 4$, $16b^2 - a^2$ by $4b + a$, $m^2n^2 - 25$ by $mn + 5$, $1 + 27x^3$ by $3x + 1$.

FACTORING.

7. Resolve each of the following expressions into its prime factors:

1. $x^2 - 1$. *Ans.* $(x + 1)\,(x - 1)$.
2. $a^2 - 9x^2$. *Ans.* $(a + 3x)\,(a - 3x)$.
3. $4x^2 - 9$. *Ans.* $(2x - 3)\,(2x + 3)$.
4. $16x^4 - 9y^2z^2$. *Ans.* $(4x^2 + 3yz)\,(4x^2 - 3yz)$.
5. $25b^2 - 4a^2$. *Ans.* $(5b + 2a)\,(5b - 2a)$.
6. $a^2 - 2ax + x^2$. *Ans.* $(a - x)\,(a - x)$.
7. $1 + 9x^2 - 6x$. *Ans.* $(1 - 3x)\,(1 - 3x)$.
8. $a^3 + x^3$. *Ans.* $(a + x)\,(a^2 - ax + x^2)$.
9. $a^4 - x^4$. *Ans.* $(a^2 + x^2)\,(a + x)\,(a - x)$.
10. $y^3 - x^2y$. *Ans.* $y\,(y + x)\,(y - x)$.
11. $7a^3 - 7x^3$. *Ans.* $7\,(a - x)\,(a^2 + ax + x^2)$.
12. $32x^5 + 243y^5$.

Ans. $(2x + 3y)\,(16x^4 - 24x^3y + 36x^2y^2 - 54xy^3 + 81y^4)$.

13. $8x^2 + 14x + 3$. *Ans.* $(4x + 1)(2x + 3)$.

14. $a^6 - x^6$.
Ans. $(a + x)(a - x)(a^2 + ax + x^2)(a^2 - ax + x^2)$.

15. $x^4 - 4a^2x^2 + 4a^4$. *Ans.* $(x^2 - 2a^2)(x^2 - 2a^2)$.

16. $x^2 + 4x + 3$. *Ans.* $(x + 1)(x + 3)$.

17. $(a + b)^2 - c^2$. *Ans.* $(a + b + c)(a + b - c)$.

18. $4a^3(a^3 - x^3)$.
Ans. $2 \cdot 2 \cdot a \cdot a \cdot a(a - x)(a^2 + ax + x^2)$.

19. $(x + 5)^2 - (x - 3)^2$. *Ans.* $2 \cdot 2 \cdot 2 \cdot 2(x + 1)$.

20. $a^2 - (b - c)^2$. *Ans.* $(a + b - c)(a - b + c)$.

21. $x^8 - a^8$. *Ans.* $(x - a)(x + a)(x^2 + a^2)(x^4 + a^4)$.

22. $x^4 + 5x^2 + 4$. *Ans.* $(x^2 + 1)(x^2 + 4)$.

23. $x^2 - 2x - 3$. *Ans.* $(x + 1)(x - 3)$.

24. $x^2 - 8x + 7$. *Ans.* $(x - 7)(x - 1)$.

25. $7x^2 - 12x + 5$. *Ans.* $(7x - 5)(x - 1)$.

26. $6x^2 + x - 1$. *Ans.* $(3x - 1)(2x + 1)$.

27. $4b^2 - 5ab + a^2$. *Ans.* $(4b - a)(b - a)$.

28. $(a + b)^2 - (c + d)^2$.
Ans. $(a + b + c + d)(a + b - c - d)$.

29. $a^2 + b^2 - c^2 + 2ab$. *Ans.* $(a + b + c)(a + b - c)$.

30. $a^2 - b^2 - c^2 + 2bc$. *Ans.* $(a + b - c)(a - b + c)$.

31. $(a - b)(b - c) + (a - d)(c - d)$.
Ans. $(a - b + c - d)(b - d)$.

32. $4a^2 - b^2 - 9c^2 + d^2 + 2(2ad - 3bc)$.
Ans. $(2a + b + 3c + d)(2a - b - 3c + d)$.

GREATEST COMMON DIVISOR.

8. Find the G. C. D. of

1. $6a^2xy$ and $9ax^2y$. *Ans.* $3axy$.
2. $85ax^2y^2z^2$ and $35a^3x^3yz$. *Ans.* $5ax^2yz$.
3. $7a^2 - 14ab$ and $3ac - 6bc$. *Ans.* $a - 2b$.
4. $x^2 - a^2$ and $x^3 + a^3$. *Ans.* $x + a$.
5. $(x + 1)^2$ and $x^2 - x - 2$. *Ans.* $x + 1$.
6. $ax(x^3 - a^3)$ and $a^2x - a^3$. *Ans.* $a(x - a)$.
7. $x^2 + 2cx$ and $3ax + 6ac$. *Ans.* $x + 2c$.
8. $6(a + x)$ and $12(a^2 + x^2)$. *Ans.* 6.
9. $35ax^2$ and $14x^3 + 7ax + 21x^2$. *Ans.* $7x$.
10. $12a^2 + 22ax + 6x^2$ and $6a^2 + 7ax - 3x^2$. *Ans.* $2a + 3x$.
11. $16a^2b^2 - 20ab^3 + 4b^4$ and $18a^4b - 18a^3b^2 + 6a^2b^3 -$ *Ans.* $2ab - 2b^2$.
12. $2x^3 - 15x + 14$ and $x^4 - 15x^2 + 28x - 12$. *Ans.* $x - 2$.
13. $x^3 - (a^2 + b^2)x + ab^2$ and $x^3 - 2ax^2 + (a^2 + b^2)x - ab^2$. *Ans.* $x - a$.
14. $3x^3 - (3c + d - 3)x - 3c - d$ and $2x^2 + (2a + b + 2)x + 2a + b$. *Ans.* $x + 1$.
15. $x^2 + x - 30$ and $x^2 + 11x + 30$. *Ans.* $x + 6$.
16. $x^3 - a^3$ and $x^2 - a^2$. *Ans.* $x - a$.
17. $6x^2 + 7x - 20$ and $3x^2 - x - 4$. *Ans.* $3x - 4$.

18. $x^5 - xy^4$ and $x^2 - y^2$. *Ans.* $x^2 - y^2$.

19. $x^3 + 3x^2 - 4x$ and $x^2 - 7x + 6$. *Ans.* $x - 1$.

20. $x^2 + 11x + 30$ and $9x^3 + 53x^2 - 9x - 18$.
Ans. $x + 6$.

21. $x^3 + 4x^2 - 5$ and $x^3 - 3x + 2$. *Ans.* $x - 1$.

22. $x^3 + 2x^2 - 3x$ and $2x^3 + 5x^2 - 3x$.
Ans. $x^2 + 3x$.

23. $14 - 34x + 12x^2$ and $42a - 4ax - 6ax^2$.
Ans. $14 - 6x$.

24. $x^3 + 2x^2 + 2x + 1$ and $x^3 - 2x - 1$.
Ans. $x + 1$.

25. $3x^3 + x^2 - 5x + 2$ and $15x^2 + 11x - 14$.
Ans. $3x - 2$.

26. $a^2 - 5ab + 4b^2$ and $a^3 - a^2b + 3ab^2 - 3b^3$.
Ans. $a - b$.

27. $x^3 - 3x^2 + 7x - 21$ and $2x^4 + 19x^2 + 35$.
Ans. $x^2 + 7$.

28. $a^3 + 2a^2b - ab^2 - 2b^3$ and $a^3 - 2a^2b - ab^2 + 2b^3$.
Ans. $a^2 - b^2$.

29. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ and $a^2 - b^2 - c^2 - 2bc$.
Ans. $a + b + c$.

30. $6x^3 - 6xy^2 + 2x^2y - 2y^3$ and $12x^2 - 15xy + 3y^2$.
Ans. $x - y$.

31. $x^4 + 7x^3 + 7x^2 - 15x$ and $x^3 - 2x^2 - 13x + 110$.
Ans. $x + 5$.

32. $x^4 - a^4$, $x^3 + a^3$, and $x^2 - a^2$. *Ans.* $x + a$.

33. $a^3 + a^2b - ab^2 - b^3$, $a^3 - 3ab^2 + 2b^3$, and $a^3 - 2a^2b - ab^2 + 2b^3$. *Ans.* $a - b$.

34. $x^2 - 2a^2 - ax$, $x^2 - 4a^2$, $x^2 - 6a^2 + ax$, and $x^2 - 8a^2 + 2ax$. *Ans.* $x - 2a$.

35. $a^5 + b^5$, $a^4 - b^4$, and $a^3 + b^3$. *Ans.* $a + b$.

36. $3x^3 - 7x^2y + 5xy^2 - y^3$, $x^2y + 3xy^2 - 3x^3 - y^3$, and $3x^3 + 5x^2y + xy^2 - y^3$. *Ans.* $3x - y$.

LEAST COMMON MULTIPLE.

9. Find the L. C. M. of

1. $6a^2$ and $9a^4$. *Ans.* $18a^4$.

2. $12x^2y^3z^4$ and $8y^5z^3x^2$. *Ans.* $24x^2y^5z^4$.

3. $x^3 - a^3$ and $(x - a)^2$.
Ans. $x^3 - ax^2 - a^2x + a^3$.

4. $a^3 - x^3$ and $a^2 - x^2$. *Ans.* $a^4 + a^3x - ax^3 - x^4$.

5. $(a^2x - ax^2)^2$ and $ax(a^2 - x^2)^2$.
Ans. $a^2x^2(a^2 - x^2)^2$.

6. $ax + ay$ and $ax - ay$. *Ans.* $a(x^2 - y^2)$.

7. $a - x$ and $a^3 - x^3$. *Ans.* $a^3 - x^3$.

8. $3(a + x)$ and $4(a^2 - x^2)$. *Ans.* $12(a^2 - x^2)$.

9. $a^2bx - ab^2y$ and $abx + b^2y$.
Ans. $ab(a^2x^2 - b^2y^2)$.

10. $x^2 + xy$ and $(x + y)^2$. *Ans.* $x^3 + 2x^2y + xy^2$.

11. b^2 and $b(b + ax)$. *Ans.* $b^3 + ab^2x$.

12. $6x^2 + 5x - 6$ and $6x^2 - 13x + 6$.

Ans. $12x^3 - 8x^2 - 27x + 18$.

13. $x^2 - 1$ and $x^2 + 4x + 3$.

Ans. $x^3 + 3x^2 - x - 3$.

14. $x^3 - x^2y$ and $x^2 - y^2$. *Ans.* $x^4 - x^2y^2$.

15. $12x^3 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$.

Ans. $36x^3 - 3ax^2 - 50a^2x + 24a^3$.

16. $x^3 - 1$ and $x^3 + 1$. *Ans.* $x^4 - x^3 + x - 1$.

17. $6x^2 + 13x + 6$ and $8x^2 + 6x - 9$.

Ans. $24x^3 + 34x^2 - 15x - 18$.

18. $x^3 - 7x^2 + 6x$ and $x^2 + 2x - 3$.

Ans. $x^4 - 4x^3 - 15x^2 + 18x$.

19. $(x + 1)^2$ and $x^3 + 1$. *Ans.* $x^4 + x^3 + x + 1$.

20. $x^3 + bx^2 + ax + ab$ and $x^2 - (a - b)\,x - ab$.

Ans. $x^4 + (b - a)x^3 + (a - ab)x^2 + (ab - a^2)x - a^2b$.

21. $x - 1$, $x^2 - 1$, and $x^2 - 2x + 1$.

Ans. $x^3 - x^2 - x + 1$.

22. $x^2 - y^2$, $(x - y)^2$, and $x^3 - y^3$.

Ans. $x^5 - x^3y^2 - x^2y^3 + y^5$.

23. $a + b$, $a - b$, and $a^2 - b^2$. *Ans.* $a^2 - b^2$.

24. $6x$, $2\,(x + 1)$, and $2\,(x + 2)$.

Ans. $6x^3 + 18x^2 + 12x$.

25. a, $a + 2b$, and $a^2 - 4b^2$. *Ans.* $a^3 - 4ab^2$.

26. x, $2x - 1$, and $4x^2 - 1$. *Ans.* $4x^3 - x$.

27. $x + y$, $2x$, and $2x\,(x - y)$. *Ans.* $2x^3 - 2xy^2$.

28. x^3, x, $(x^2+1)^2$, and x^3+1.

Ans. $x^3(x^2+1)^2$.

29. $x-1$, x^2+x+1, and x^3-1. *Ans.* x^3-1.

30. b, $a-b$, and a^2b-b^3. *Ans.* a^2b-b^3.

31. $6x^2-13x+6$, $6x^2+5x-6$, and $9x^2-4$.

Ans. $(4x^2-9)(9x^2-4)$.

32. x, a, $x(a+x)$, and $a(x-a)$.

Ans. $ax(x^2-a^2)$.

33. $a-b+c$, $a+b-c$, $c-a+b$, $c^2-(a-b)^2$, and $a^2-(b-c)^2$.

Ans. $(a+c-b)(a+b-c)(b+c-a)$.

FRACTIONS.

10.—1. Multiply $\frac{5a}{12}$ by x. *Ans.* $\frac{5ax}{12}$.

2. Multiply $\frac{4x+1}{6x}$ by 3. *Ans.* $\frac{4x+1}{2x}$.

3. Multiply $\frac{x^2-a^2}{ax}$ by ax. *Ans.* x^2-a^2.

4. Multiply $\frac{a+b}{c+d}$ by $a-b$. *Ans.* $\frac{a^2-b^2}{c+d}$.

5. Multiply $\frac{m+n}{a^2-b^2}$ by $a+b$. *Ans.* $\frac{m+n}{a-b}$.

6. Multiply $\frac{x-a}{x^3-x}$ by $x+1$. *Ans.* $\frac{x-a}{x^2-x}$.

11.—1. Divide $\frac{4x+2}{3}$ by $2x+1$. *Ans.* $\frac{2}{3}$.

2. Divide $\frac{a^2 - b^2}{a + 2b}$ by $a - b$. *Ans.* $\frac{a + b}{a + 2b}$.

3. Divide $\frac{m + n}{a - b}$ by $a + b$. *Ans.* $\frac{m + n}{a^2 - b^2}$.

4. Divide $\frac{a}{x^2 - xy + y^2}$ by $x + y$. *Ans.* $\frac{a}{x^3 + y^3}$.

5. Divide $\frac{x^2 + 2xy + y^2}{a + b}$ by $x + y$. *Ans.* $\frac{x + y}{a + b}$.

6. Divide $\frac{x^3 - y^3}{m - n}$ by $x - y$. *Ans.* $\frac{x^2 + xy + y^2}{m - n}$.

REDUCTION OF FRACTIONS.

12. Reduce each of the following fractions to its lowest terms:

1. $\frac{9ax}{15a^2}$. *Ans.* $\frac{3x}{5a}$.

2. $\frac{5ax^2}{35bx^3}$. *Ans.* $\frac{a}{7bx}$.

3. $\frac{3abx^2}{6ax}$. *Ans.* $\frac{bx}{2}$.

4. $\frac{14x^2y^2}{7x^3y}$. *Ans.* $\frac{2y}{x}$.

5. $\frac{96a^2x^2y^2z^2}{26a^2x^3yz}$. *Ans.* $\frac{48yz}{13x}$.

6. $\frac{51x^3 - 17x^2 + 34x}{17x^6}$. *Ans.* $\frac{3x^2 - x + 2}{x^5}$.

7. $\frac{6x^2yz + 21xyz - 18x^2y^2z^2}{6axy}$. *Ans.* $\frac{2xz + 7z - 6xyz^2}{2a}$.

8. $\dfrac{ax^2 + x^2}{3cx - dx}$. *Ans.* $\dfrac{ax + x}{3c - d}$.

9. $\dfrac{14x^3y^2 - 21x^2y^3}{42x^3y^3}$. *Ans.* $\dfrac{2x - 3y}{6xy}$.

10. $\dfrac{3a^2 + 3ax}{a^2 - x^2}$. *Ans.* $\dfrac{3a}{a - x}$.

11. $\dfrac{a - b}{a^3 - b^3}$. *Ans.* $\dfrac{1}{a^2 + ab + b^2}$.

12. $\dfrac{x^4 - b^4}{x^5 - b^2x^3}$. *Ans.* $\dfrac{x^2 + b^2}{x^3}$.

13. $\dfrac{2x^3 + 5x^2 - 3x}{x^3 + 2x^2 - 3x}$. *Ans.* $\dfrac{2x - 1}{x - 1}$.

14. $\dfrac{a^3 + 2a^2}{a^2 + 4a + 4}$. *Ans.* $\dfrac{a^2}{a + 2}$.

15. $\dfrac{3ax^2 - 13ax + 14a}{7x^2 - 17x^2 + 6x}$. *Ans.* $\dfrac{3ax - 7a}{7x^2 - 3x}$.

16. $\dfrac{10x - 24x^2 + 14x^3}{15 - 24x + 3x^2 + 6x^3}$. *Ans.* $\dfrac{14x^2 - 10x}{6x^2 + 9x - 15}$.

17. $\dfrac{x^3 - 9x^2 + 23x - 15}{x^2 - 8x + 7}$. *Ans.* $\dfrac{x^2 - 8x + 15}{x - 7}$.

18. $\dfrac{2ax^3 + ax^2 - 8ax + 5a}{7x^3 - 12x^2 + 5x}$. *Ans.* $\dfrac{2ax^2 + 3ax - 5a}{7x^2 - 5x}$.

19. $\dfrac{x^3 - 3x^2 + 3x - 2}{x^3 - 4x^2 + 6x - 4}$. *Ans.* $\dfrac{x^2 - x + 1}{x^2 - 2x + 2}$.

20. $\dfrac{12\,(a^4 - x^4)}{15\,(x^3 - a^3)}$. *Ans.* $-\dfrac{4\,(a^3 + a^2x + ax^2 + x^3)}{5\,(a^3 + ax + x^2)}$.

21. $\dfrac{x^2 - (a - b)\,x - ab}{x^3 + bx^2 + ax + ab}$. *Ans.* $\dfrac{x - a}{x^2 + a}$.

22. $\dfrac{x^4 + (2b^2 - a^2)\,x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$. *Ans.* $\dfrac{x^2 - ax + b^2}{x^2 + ax - b^2}$.

23. $\dfrac{a^3 + (1 + a)\,ab + b^2}{a^4 - b^2}$. *Ans.* $\dfrac{a + b}{a^2 - b}$.

24. $\dfrac{x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4}{x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4}$. *Ans.* $\dfrac{x + 4a}{x + 2a}$.

25. $\dfrac{a^5 + 3ab^4 - 4a^2b^3}{a^4 - a^3b - ab^3 + b^4}$. *Ans.* $\dfrac{a^4 + a^3b + a^2b^2 - 3ab^3}{a^3 - b^3}$.

26. $\dfrac{15x^3 + 35x^2 + 3x + 7}{27x^4 + 63x^3 - 12x^2 - 28x}$. *Ans.* $\dfrac{5x^2 + 1}{9x^3 - 4x}$.

27. $\dfrac{3x^2 - (4a + 2b)\,x + 2ab + a^2}{x^3 - (2a + b)\,x^2 + (2ab + a^2)\,x - a^2b}$.

Ans. $\dfrac{3x - 2b - a}{x^2 - (a + b)\,x + ab}$.

28. $\dfrac{ax^m - bx^{m+1}}{a^2bx - b^3x^2}$. *Ans.* $\dfrac{ax^{m-1} - bx^{m-1}}{a^2b - b^3x^2}$.

13.—1. Reduce $\dfrac{ax^2 + 2x^2}{a + 2}$ to a mixed quantity.

Ans. $x + \dfrac{x^2}{a + 2}$.

2. Reduce $\dfrac{a^2 - x^2}{a}$ to a mixed quantity. *Ans.* $a - \dfrac{x^2}{a}$.

3. Reduce $\dfrac{6x + 2y}{2x}$ to a mixed quantity. *Ans.* $3 + \dfrac{y}{x}$.

4. Reduce $\dfrac{25x^3 - 3a + 2c}{5x}$ to a mixed quantity.

Ans. $5x^2 - \dfrac{3a - 2c}{5x}$.

5. Reduce $\frac{x^3 - y^3}{x - y}$ to an entire quantity.

Ans. $x^2 + xy + y^2$.

6. Reduce $\frac{ax + 2x^2}{a + x}$ to a mixed quantity.

Ans. $x + \frac{x^2}{a + x}$.

7. Reduce $\frac{a^2 + 3x^2}{a + x}$ to a mixed quantity.

Ans. $a - x + \frac{4x^2}{a + x}$.

8. Reduce $\frac{a^4 - x^4}{a^2 + x^2}$ to an entire quantity.

Ans. $a^2 - x^2$.

9. Reduce $\frac{a^4}{a^2 - x^2}$ to a mixed quantity.

Ans. $a^2 + x^2 + \frac{x^4}{a^2 - x^2}$.

10. Reduce $\frac{x^2 + 6x + 12y - 5y^2}{x - 2y + 6}$ to a mixed quantity.

Ans. $x + 2y - \frac{y^2}{x - 2y + 6}$.

11. Reduce $\frac{x^2 + x}{x - 1}$ to a mixed quantity.

Ans. $x + 2 + \frac{2}{x - 1}$.

12. Reduce $\frac{4ax - 2x^2 - a^2}{2a - x}$ to a mixed quantity.

Ans. $2x - \frac{a^2}{2a - x}$.

13. Reduce $\frac{2(a^3 + x^3)}{a^2 + x^2}$ to a mixed quantity.

Ans. $2a - \frac{2ax^2 - 2x^3}{a^2 + x^2}$.

14.—1. Reduce $a + b$ to a fraction whose denominator shall be $a - b$.

Ans. $\frac{a^2 - b^2}{a - b}$.

2. Reduce $x^2 + xy + y^2$ to a fraction whose denominator shall be $x - y$.

Ans. $\frac{x^3 - y^3}{x - y}$.

3. Reduce $a^2 - x^2$ to a fraction whose denominator shall be $a^2 + x^2$.

Ans. $\frac{a^4 - x^4}{a^2 + x^2}$.

4. Reduce $(a + b)^2$ to a fraction whose denominator shall be $a + b$.

Ans. $\frac{(a + b)^3}{(a + b)}$.

5. Reduce $(a - b)^2$ to a fraction whose denominator shall be $(a + b)^2$.

Ans. $\frac{(a^2 - b^2)^2}{(a + b)^2}$.

6. Rednce a^m to a fraction whose denominator shall be b^n

Ans. $\frac{a^m b^n}{b^n}$.

15.—1. Reduce $\frac{a}{b}$ to a fraction whose denominator shall be bdf.

Ans. $\frac{adf}{bdf}$.

2. Reduce $\frac{3x}{5a}$ to a fraction whose denominator shall be $15a$.

3. Reduce $\frac{2}{2x - 3}$ to a fraction whose denominator shall be $16x^4 - 81$.

Ans. $\frac{16x^3 + 24x^2 + 36x + 54}{16x^4 - 81}$.

4. Reduce each of the fractions $\frac{2a}{3}$, $\frac{3b}{5}$, $\frac{4c}{7}$ to an equivalent one whose denominator shall be 105.

Ans. $\frac{70a}{105}$, $\frac{63b}{105}$, $\frac{60c}{105}$.

5. Reduce each of the fractions $\frac{x}{a^2+x^2}$, $\frac{x}{a^2-x^2}$ to an equivalent one whose denominator shall be a^4-x^4.

Ans. $\frac{a^2x-x^3}{a^4-x^4}$, $\frac{a^2x+x^3}{a^4-x^4}$.

6. Reduce each of the fractions $\frac{x+1}{x-1}$, $\frac{x^2+1}{x^2+x+1}$ to an equivalent one whose denominator shall be x^3-1.

Ans. $\frac{x^3+2x^2+2x+1}{x^3-1}$, $\frac{x^3-x^2+x-1}{x^3-1}$.

16. Reduce each of the following expressions to the form of a fraction:

1. $3a+\frac{5a}{6x^2}$. *Ans.* $\frac{18ax^2+5a}{6x^2}$.

2. $5x-\frac{2x-3}{7}$. *Ans.* $\frac{33x+3}{7}$.

3. $3a+\frac{b}{c}$. *Ans.* $\frac{3ac+b}{c}$.

4. $2a+\frac{ax-a}{x}$. *Ans.* $\frac{3ax-a}{x}$.

5. $5ax-\frac{ax^2-a^3}{x}$. *Ans.* $\frac{4ax^2+a^3}{x}$.

6. $1+\frac{x}{1-x}$. *Ans.* $\frac{1}{1-x}$.

7. $2a+\frac{4b^2}{a-b}$. *Ans.* $\frac{2a^2-2ab+4b^2}{a-b}$.

8. $a+x+\frac{a^2+x^2}{a-x}$. *Ans.* $\frac{2a^2}{a-x}$.

9. $3 + \frac{3}{x^2-1}$. *Ans.* $\frac{3x^2}{x^2-1}$.

10. $ax + 4 - \frac{ax^2-y}{x+y}$. *Ans.* $\frac{axy+4x+5y}{x+y}$.

11. $\frac{a+b+c}{a+c} - 1$. *Ans.* $\frac{b}{a+c}$.

12. $a^2 - ax + x^2 + \frac{2x^3}{a+x}$. *Ans.* $\frac{a^3+3x^3}{a+x}$.

13. $1 - \frac{a^2-2ax+x^2}{a^2+x^2}$. *Ans.* $\frac{2ax}{a^2+x^2}$.

14. $a^2 + ab + b^2 - \frac{a^3+b^3}{a-b}$. *Ans.* $\frac{2b^3}{b-a}$.

17. Reduce each of the following sets of fractions to equivalent fractions having a common denominator:

1. $\frac{2a}{3}, \frac{3b}{5}, \frac{4c}{7}$. *Ans.* $\frac{70a}{105}, \frac{63b}{105}, \frac{60c}{105}$.

2. $\frac{x}{y}, \frac{y}{x}$. *Ans.* $\frac{x^2}{xy}, \frac{y^2}{xy}$.

3. $\frac{a}{b}, \frac{c}{y}, \frac{d}{z}$. *Ans.* $\frac{ayz}{byz}, \frac{bcz}{byz}, \frac{bdy}{byz}$.

4. $\frac{x}{a-b}, \frac{x}{a+b}$. *Ans.* $\frac{ax+bx}{a^2-b^2}, \frac{ax-bx}{a^2-b^2}$.

5. $\frac{5x}{a+x}, \frac{a-x}{2}, \frac{1}{3x}$.

Ans. $\frac{30x^2}{6ax+6x^2}, \frac{3a^2x-3x^3}{6ax+6x^2}, \frac{2a+2x}{6ax+6x^2}$.

18. Reduce each of the following sets of fractions to equivalent fractions having the least common denominator:

1. $\frac{a}{3}, \frac{2a}{5}, \frac{7a}{10}$. *Ans.* $\frac{10a}{30}, \frac{12a}{30}, \frac{21a}{30}$.

2. $\frac{x}{ab}, \frac{y}{ac}, \frac{z}{bc}$. *Ans.* $\frac{cx}{abc}, \frac{by}{abc}, \frac{az}{abc}$.

3. $\frac{2}{ab}, \frac{3}{bc}, \frac{4}{a^2}$. *Ans.* $\frac{2ac}{a^2bc}, \frac{3a^2}{a^2bc}, \frac{4bc}{a^2bc}$.

4. $\frac{1}{a-x}, \frac{1}{a^2-x^2}, \frac{1}{a+x}$.

Ans. $\frac{a+x}{a^2-x^2}, \frac{1}{a^2-x^2}, \frac{a-x}{a^2-x^2}$.

5. $\frac{2}{ab}, \frac{3}{2bc}, \frac{4}{3cd}, \frac{5}{4de}, \frac{6}{5ef}$.

Ans. $\frac{120cdef}{60abcdef}, \frac{90adef}{60abcdef}, \frac{80abef}{60abcdef}, \frac{75abcf}{60abcdef}, \frac{72abcd}{60abcdef}$.

6. $\frac{x+1}{x-1}, \frac{x^2+1}{x^2+x+1}, \frac{x^3}{x^3-1}$.

Ans. $\frac{x^3+2x^2+2x+1}{x^3-1}, \frac{x^3-x^2+x-1}{x^3-1}, \frac{x^3}{x^3-1}$.

COMBINATIONS OF FRACTIONS.

19. Perform the additions indicated in the following expressions:

1. $\frac{3a}{5}+\frac{2a}{7}+\frac{a}{3}$. *Ans.* $\frac{128a}{105}$.

2. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$. *Ans.* $\frac{2(a^2+b^2)}{a^2-b^2}$.

3. $\frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}$. *Ans.* 0.

4. $\frac{x}{7}+\frac{5x}{9}+\frac{4x}{11}$. *Ans.* $\frac{736x}{693}$.

5. $4x+\frac{7x}{9}+2+\frac{x}{5}$. *Ans.* $\frac{224x+90}{45}$.

6. $\frac{5a^2+b}{5}+\frac{4a^2+2b^2}{5b}$. *Ans.* $\frac{5a^2b+4a^2+3b^2}{5b}$.

7. $\frac{2x+1}{3x}+\frac{4x+2}{5x}+\frac{1}{7}$. *Ans.* $\frac{169x+77}{105x}$.

8. $\frac{3a^2}{b}+\frac{2a}{5}+\frac{3b}{7a}$. *Ans.* $\frac{105a^3+14a^2b+15b^2}{35ab}$.

9. $\frac{x}{a+b}+\frac{x}{a-b}$. *Ans.* $\frac{2ax}{a^2-b^2}$.

10. $\frac{a}{b}+\frac{a+b}{a-b}$. *Ans.* $\frac{a^2+b^2}{ab-b^2}$.

11. $\frac{x}{x-3}+\frac{x}{2-x}$. *Ans.* $\frac{x}{x^2-5x+6}$.

12. $\frac{x-a}{a}+\frac{2x-a}{x-a}$. *Ans.* $\frac{x^2}{ax-a^2}$.

13. $\frac{1+x}{1+x+x^2}+\frac{1-x}{1-x+x^2}$. *Ans.* $\frac{2}{1+x^2+x^4}$.

14. $\frac{a}{b}+\frac{-ac}{b(b+c)}$. *Ans.* $\frac{a}{b+c}$.

15. $\frac{a}{b}+\frac{c(bd-ae)}{b(ce-bf)}$. *Ans.* $\frac{cd-af}{ce-bf}$.

16. $\frac{x-1}{x+1}+\frac{x+1}{x-1}+\frac{3x-2}{x+3}+\frac{3x-4}{x-2}$.

Ans. $\frac{8x^4-x^3-24x^2+5x-4}{(x^2-1)(x+3)(x-2)}$.

17. $\frac{a+x}{a-x}+\frac{a-x}{a+x}+\frac{a^2+x^2}{a^2-x^2}+\frac{4ax}{a^2+x^2}$.

Ans. $\frac{3a^4+4a^3x+6a^2x^2-4ax^3+3x^4}{a^4-x^4}$.

18. $\frac{1}{x^2-x+1}+\frac{1}{x^2+x+1}+\frac{1}{x^4-x^2+1}$.

Ans. $\frac{2x^6+x^4+x^2+3}{x^6+x^4+1}$.

19. $\frac{1}{x-1}+\frac{1}{x+1}+\frac{4x-8}{2-2x^2}$. *Ans.* $\frac{4}{x^2-1}$.

20. Perform the subtractions indicated in the following expressions:

1. $\frac{7x}{2}-\frac{2x-1}{3}$. *Ans.* $\frac{17x+2}{6}$.

2. $\frac{1}{x-y}-\frac{1}{x+y}$. *Ans.* $\frac{2y}{x^2-y^2}$.

3. $\frac{x}{3}-\frac{2x}{7}$. *Ans.* $\frac{x}{21}$.

4. $\frac{3x}{7}-\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

5. $\frac{2a-b}{4c}-\frac{3a-4b}{3b}$. *Ans.* $\frac{6ab-3b^2-12ac+16bc}{12bc}$.

6. $3a + \frac{11a - 10}{15} - \left(2a + \frac{3a - 5}{7}\right)$.

Ans. $a + \frac{32a + 5}{105}$.

7. $x + \frac{x - y}{x^2 + xy} - \frac{x + y}{x^2 - xy}$. *Ans.* $x - \frac{4y}{x^2 - y^2}$.

8. $\frac{a - b}{2c} - \frac{2b - 4a}{5d}$. *Ans.* $\frac{5ad - 5bd - 4bc + 8ac}{10cd}$.

9. $3x + \frac{x}{b} - \left(x - \frac{x - a}{c}\right)$. *Ans.* $2x + \frac{cx + bx - ab}{bc}$.

10. $\frac{a + b}{a - b} - \frac{a - b}{a + b}$. *Ans.* $\frac{4ab}{a^2 - b^2}$.

11. $\frac{(x + y)^2}{xy} - \frac{(x - y)^2}{xy}$. *Ans.* 4.

12. $\frac{1}{(a - b)(a - c)} - \frac{1}{(a - c)(c - b)}$.

Ans. $\frac{1}{(a - b)(b - c)}$.

13. $\frac{1}{1 + x} - \left[\frac{6}{1 - x} - \left(\frac{2}{1 + 2x} - \frac{16}{2x - 1}\right)\right]$.

Ans. $\frac{2x^2 + 21x + 13}{(1 - x^2)(1 - 4x^2)}$.

14. $\frac{1}{(n + 1)(n + 2)} - \left[\frac{1}{(n + 1)(n + 2)(n + 3)} + \frac{1}{(n + 1)(n + 3)}\right]$. *Ans.* 0.

15. $\frac{2}{x - 1} - \left(\frac{1}{x + 1} + \frac{x}{x^2 + 1} + \frac{3}{x^3 - 1}\right)$. *Ans.* $\frac{2x}{x^4 - 1}$.

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21. Perform the multiplications indicated in the following expressions:

1. $\frac{5a}{12} \times x \times \frac{4y}{7}$. *Ans.* $\frac{5axy}{21}$.

2. $\frac{a+b}{a} \times \frac{2a^2}{a-b}$. *Ans.* $\frac{2a(a+b)}{a-b}$.

3. $\frac{a^2-b^2}{4} \times \frac{a+b}{(a-b)^2}$. *Ans.* $\frac{(a+b)^2}{4(a-b)}$.

4. $\left(\frac{x}{a}-\frac{a}{x}\right)\left[\frac{ax}{(a+x)^2}\right]$. *Ans.* $\frac{x-a}{x+a}$.

5. $\frac{1-x^2}{1+y} \times \frac{1-y^2}{x+x^2}\left(1+\frac{x}{1-x}\right)$. *Ans.* $\frac{1-y}{x}$.

6. $\frac{a^3-b^3}{a^3+b^3} \times \frac{(a+b)^2}{(a-b)^2}$. *Ans.* $\frac{a^3+2a^2b+2ab^2+b^3}{a^3-2a^2b+2ab^2-b^3}$.

7. $\frac{a}{b^2} \times \frac{b^2}{c^3} \times \frac{c^3}{d^4}$. *Ans.* $\frac{a}{d^4}$.

8. $\frac{2a}{3(a-b)} \times \frac{6(a^2-b^2)}{7}$. *Ans.* $\frac{4a^2+4ab}{7}$.

9. $\frac{3x^2}{5x-10} \times \frac{30x-60}{4x}$. *Ans.* $\frac{9x}{2}$.

10. $\frac{3x^2-x}{5} \times \frac{10}{2x^2-4x}$. *Ans.* $\frac{3x-1}{x-2}$.

11. $\frac{(a-b)^2}{a+b} \times \frac{b}{a(a-b)}$. *Ans.* $\frac{ab-b^2}{ab+a^2}$.

12. $\frac{a^4-b^4}{a^2-2ab+b^2} \times \frac{a-b}{a^2+ab}$. *Ans.* $\frac{a^2+b^2}{a}$.

13. $\left(1 - \frac{a-b}{a+b}\right)\left(2 + \frac{2b}{a-b}\right)$. *Ans.* $\frac{4ab}{a^2-b^2}$.

14. $\frac{a^2-x^2}{a+b} \times \frac{a^2-b^2}{ax+x^2} \times \left(a + \frac{ax}{a-x}\right)$. *Ans.* $\frac{a^3-a^2b}{x}$.

15. $\frac{m^2-mn+n^2}{m^3-3mn(m-n)-n^3} \times \frac{m^2-n^2}{m^3+n^3}$.

Ans. $\frac{1}{(m-n)^2}$.

22. Perform the divisions indicated in the following expressions:

1. $\frac{a+b}{c} \div \frac{c}{a+b}$. *Ans.* $\frac{(a+b)^2}{c^2}$.

2. $\frac{5x}{a} \div \frac{b}{c}$. *Ans.* $\frac{5cx}{ab}$.

3. $\frac{15ab}{a-x} \div \frac{10ac}{a^2-x^2}$. *Ans.* $\frac{3b(a+x)}{2c}$.

4. $\frac{2x^2-7}{x+a} \div \frac{a^2}{x^2+2ax+a^2}$. *Ans.* $\frac{(2x^2-7)(x+a)}{a^2}$.

5. $\frac{x^4-b^4}{(x-b)^2} \div \frac{x+b}{x-b}$. *Ans.* x^2+b^2.

6. $\frac{2ax+x^2}{a^3-x^3} \div \frac{x}{a-x}$. *Ans.* $\frac{2a+x}{a^2+ax+x^2}$.

7. $\frac{14x-3}{5} \div \frac{10x-4}{25}$. *Ans.* $\frac{70x-15}{10x-4}$.

8. $\frac{9x^2-3x}{5} \div \frac{x^2}{5}$. *Ans.* $\frac{9x-3}{x}$.

9. $\dfrac{6x-7}{x+1} \div \dfrac{x-1}{3}$. *Ans.* $\dfrac{18x-21}{x^2-1}$.

10. $\dfrac{x+x^2}{3a^2} \div \dfrac{2ax+2ax^2}{7}$. *Ans.* $\dfrac{7}{6a^2}$.

11. $\dfrac{a^6-x^3}{a^2-2ax+x^2} \div \dfrac{a^2+ax+x^2}{a-x}$. *Ans.* a^3+x^3.

12. $\dfrac{9y^2-3y}{5} \div \dfrac{y^2}{5}$. *Ans.* $\dfrac{9y-3}{y}$.

13. $\dfrac{an-nx}{a+b} \div \dfrac{am-mx}{a+b}$. *Ans.* $\dfrac{n}{m}$.

14. $12 \div \left[\dfrac{(a+x)^2}{x} - a\right]$. *Ans.* $\dfrac{12x}{a^2+ax+x^2}$.

15. $\dfrac{ab+bx}{x} \div \dfrac{a}{x}$. *Ans.* $b+\dfrac{bx}{a}$.

16. $\dfrac{x-b}{6c^2x} \div \dfrac{3cx}{4d}$. *Ans.* $\dfrac{4d(x-b)}{18c^3x^2}$.

17. $a \div \left(\dfrac{x}{x+y} \times \dfrac{a}{x-y}\right)$. *Ans.* $\dfrac{x^2-y^2}{x}$.

18. $\dfrac{3(x^2-1)}{2(a+b)} \div \left(\dfrac{x+1}{2a} \times \dfrac{x-1}{a+b}\right)$. *Ans.* $3a$.

23. Simplify each of the following expressions:

1. $\dfrac{\dfrac{a}{ab}+\dfrac{b}{a+b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}$. *Ans.* $\dfrac{(a+b+b^2)(a-b)}{b(a^2+b^2)}$.

2. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$. *Ans.* $\dfrac{b}{a}$.

3. $\dfrac{a+\dfrac{b-a}{1+ab}}{1-a\dfrac{b-a}{1+ab}}$. *Ans.* b.

4. $\dfrac{\dfrac{2a+b}{a+b}-1}{1-\dfrac{b}{a+b}}$. *Ans.* 1.

5. $\dfrac{\dfrac{a^3+b^3}{a^2-b^2}}{\dfrac{a^2-ab+b^2}{a-b}}$. *Ans.* 1.

6. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}+\dfrac{\dfrac{1}{1+x}}{\dfrac{x}{1-x}}+\dfrac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}$. *Ans.* $\dfrac{3+x^2}{x(1-x^2)}$.

7. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$. *Ans.* $\dfrac{(a+b+c)^2}{2bc}$.

8. $\dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$. *Ans.* $\dfrac{4a^2x}{x^4-a^4}$.

9. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}} \div \dfrac{1}{x+\dfrac{1}{y}} - \dfrac{1}{y(xyz+x+z)}$. *Ans.* 1.

10. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$. *Ans.* $\dfrac{adf+ae}{bdf+be+cf}$.

SIGNS OF FRACTIONS.

24.—1. What is the real sign of the fraction $\dfrac{3a^2-3b^3}{5a+5b^3}$, when $a=-3$ and $b=2$? *Ans.* +.

2. What is the real sign of $\dfrac{a^3-b^3}{a^2-2ab+b^2}$, when $a=-3$ and $b=1$? *Ans.* −.

3. What is the real sign of $\dfrac{a^2b+ab^2}{3(a+b)^2}$, when $a=-3$ and $b=2$? *Ans.* +.

4. Show that the real sign of $\dfrac{a-b}{c-d}$ is contrary to that of $\dfrac{b-a}{c-d}$, and the same as that of $\dfrac{b-a}{d-c}$.

TRANSFORMATION OF EQUATIONS.

CLEARING EQUATIONS OF FRACTIONS.

25. Transform each of the following equations into another whose terms shall be entire:

1. $\dfrac{3x}{4}+12=\dfrac{5x}{6}+9$. *Ans.* $9x+144=10x+108$.

2. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=26$. *Ans.* $6x+4x+3x=312$.

3. $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}=\frac{x}{2}+17$.

Ans. $12x+15x+20x=30x+1020$.

4. $\frac{2x}{3}-1=\frac{x}{12}+\frac{4}{3}$. *Ans.* $8x-12=x+16$.

5. $13\frac{1}{4}-\frac{1}{2}x=2x-8\frac{3}{4}$. *Ans.* $55-2x=8x-35$.

6. $\frac{2x-6}{5}-\frac{x-4}{9}-\frac{3x}{13}=0$.

Ans. $234x-702-65x+260-135x=0$.

7. $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}$.

Ans. $15x-48x+48=24x-32+10x$.

8. $\frac{x-1\frac{5}{16}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{10-3x}{4}}{39}$.

Ans. $78x-153-24+72x=156x-20x+10-3x$.

9. $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left(1-\frac{x^2}{54}\right)$.

Ans. $44x^2-187x-11x+66x^2=99x^2-594+11x^2$.

TRANSPOSITION OF TERMS.

26. In each of the following equations transpose the unknown terms of the second member into the first, and the known terms of the first member into the second:

1. $2x+13=x+17$. *Ans.* $2x-x=17-13$.
2. $7x-6=6x-4$. *Ans.* $7x-6x=6-4$.

3. $4x + 2 = 3x + 9$. *Ans.* $4x - 3x = 9 - 2$.

4. $4x + 2a = 3x + 7b$. *Ans.* $4x - 3x = 7b - 2a$.

5. $x - a = (b - a)\,x$. *Ans.* $x + (a - b)\,x = a$.

6. $7x - 4 - (3x - 11) = 0$. *Ans.* $7x - 3x = 4 - 11$.

7. $2x - 5\,(1 + x) - 7 = 0$. *Ans.* $2x - 5x = 7 + 5$.

8. $x^2 + 80 = 600 - 6x$. *Ans.* $x^2 + 6x = 600 - 80$.

9. $(2x - 3)^2 = 8x$. *Ans.* $4x^2 - 12x - 8x = -9$.

10. $(a + x)\,(a - x) = b^2 - 2x$. *Ans.* $2x - x^2 = b^2 - a^2$.

SIMPLE EQUATIONS.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

27. Find the value of x in each of the following equations:

1. $3x - 4 = 7x - 16$. *Ans.* $x = 3$.

2. $3x + 9 - 1 - 5x = 0$. *Ans.* $x = 4$.

3. $4x + 7 = x + 21 + x$. *Ans.* $x = 7$.

4. $5ax - c = b - 3ax$. *Ans.* $x = \dfrac{b + c}{8a}$.

5. $ax + b = 9x + c$. *Ans.* $x = \dfrac{b - c}{9 - a}$.

6. $2x + \dfrac{3}{4}x + \dfrac{1}{3}x - 3a = 4b + 3a$.

Ans. $x = \dfrac{48b + 72a}{37}$.

7. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=39.$ *Ans.* $x=36.$

8. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=a.$ *Ans.* $x=\frac{12a}{13}.$

9. $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}.$ *Ans.* $x=9.$

10. $\frac{3x}{5}+2\frac{1}{2}+11=\frac{2x}{8}+17.$ *Ans.* $x=10.$

11. $\frac{x}{3}-5+\frac{x}{4}+8+\frac{x}{5}-10=100-6-7.$

Ans. $x=120.$

12. $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}.$ *Ans.* $x=4.$

13. $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}.$ *Ans.* $x=8.$

14. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}.$ *Ans.* $x=8.$

15. $\frac{20x}{25}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}.$ *Ans.* $x=4.$

16. $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}.$ *Ans.* $x=5.$

17. $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}.$ *Ans.* $x=9.$

18. $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}.$ *Ans.* $x=13.$

19. $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}.$ *Ans.* $x=12.$

20. $x - \frac{2x+1}{3} = \frac{x+3}{4}$. *Ans.* $x = 13$.

21. $\frac{5x+5}{x+2} + 1 = \frac{6x-12}{x-2}$. *Ans.* $x = 2$.

22. $\frac{2x+a}{b} - \frac{x-b}{a} = \frac{3ax+(a-b)^2}{ab}$.

Ans. $x = \frac{2ab}{a+b}$.

23. $\frac{x-3\frac{1}{2}}{2} - \frac{5-4x}{7} = x + 12 - \frac{3x - \left(\frac{9-2x}{3} - 2\right)}{6}$.

Ans. $x = 21\frac{11}{13}$.

24. $\frac{x - \frac{2x-36}{9}}{8} - \frac{x-18}{6} = x + 9 - \frac{5x - \frac{2(x-10)}{13}}{4}$.

Ans. $x = 36$.

PROBLEMS.

28.—1. What number is that, to the double of which if 18 be added, the sum will be 82? *Ans.* 32.

2. What number is that, to the double of which if 44 be added, the sum will be equal to four times the required number? *Ans.* 22.

3. What number is that, the double of which exceeds its half by 6? *Ans.* 4.

4. From two towns which are 187 miles distant, two travelers set out at the same time with the intention of meeting. One of them goes 8 miles, and the other 9 miles, a day. In how many days will they meet? *Ans.* 11.

5. A man distributed $60 among four beggars. To the second he gave twice, to the third three times, and to the fourth four times as much as to the first. What did he give to each? *Ans.* $6, $12, $18, $24.

6. A bookseller sold 10 books at a certain price, and afterward 15 more at the same rate. At the second sale he received $10 more than at the first. What did he receive for each book? *Ans.* $2.

7. A, B, and C together have $140. A has twice as much as B, and B three times as much as C. How much has each? *Ans.* A, $84; B, $42; C, $14.

8. Four merchants, A, B, C, and D, entered into a speculation, for which they subscribed $4755, of which B paid three times as much as A; C paid as much as A and B; and D paid as much as C and B. What did each pay?

Ans. A, $317; B, $951; C, $1268; D, $2219.

9. A tailor bought three pieces of cloth, which together contained 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each piece?

Ans. 1st, 35 yds.; 2d, 50 yds.; 3d, 74 yds.

10. A cask which held 146 gallons was filled with a mixture of brandy, wine, and water. In it there were 15 gallons more of wine than of brandy, and as many gallons of water as of wine and brandy together. How many gallons were there of each? *Ans.* 29 of brandy, 44 of wine, 73 of water.

11. A person employed four workmen, to the first of whom he gave $2 more than to the second, to the second $3 more than to the third, and to the third $4 more than to the fourth. Their wages amounted to $32. What did each receive? *Ans.* 1st, $12; 2d, $10; 3d, $7; 4th, $3.

12. What number is that whose third part added to its fourth part makes 21? *Ans.* 36.

13. A man has a lease for 20 years, and one-third of the time past is equal to one-half of the time to come. How much of the time has passed? *Ans.* 12 years.

14. What number is that, from which 6 being subtracted and the remainder multiplied by 11, the product will be 121? *Ans.* 17.

15. Find two numbers whose difference is 6, and if one-third of the less be added to one-fifth of the greater, the sum will be equal to one-third of the greater diminished by one-fifth of the less. *Ans.* 2 and 8.

16. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9. *Ans.* 12 and 36.

17. An estate is to be divided among 4 children, in the following manner:

The first is to have \$200 more than $\frac{1}{4}$ of the whole.

The second is to have \$340 more than $\frac{1}{4}$ of the whole.

The third is to have \$300 more than $\frac{1}{4}$ of the whole.

And the fourth is to have \$400 more than $\frac{1}{4}$ of the whole.

What is the value of the estate? *Ans.* \$4800.

18. In the composition of a quantity of gunpowder,

The weight of the nitre was 10 lbs. more than $\frac{2}{3}$ of the whole.

The weight of the sulphur was $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole.

The weight of the charcoal was 2 lbs. less than $\frac{1}{4}$ of the nitre.

What was the weight of the gunpowder? *Ans.* 69 lbs.

19. A man allows \$1000 per annum for the expenses of his family, and annually increases that part of his capital which is not so expended by one-third of it; at the end of three years his original capital will be doubled. What is the original capital? *Ans.* \$14800.

20. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives shall be equal to $\frac{5}{10}$ of what the second receives. *Ans.* \$63 and \$120.

21. Divide the number 68 into two such parts that the difference between the greater and 84 shall be equal to three times the difference between the less and 40. *Ans.* 42 and 26.

22. A laborer engaged to serve for 60 days on these conditions: That for every day he worked he should have 75 cents and his board, and for every day he was idle he should forfeit 25 cents for damage and board. At the end of the time a settlement was made and he received \$25. How many days did he work, and how many days was he idle? *Ans.* He worked 40 days, and was idle 20 days.

23. A boy engaged to carry 100 glass vessels to a certain place, and to receive 3 cents for every one he delivered, and to forfeit 9 cents for every one he broke. On settlement, he received 2 dollars and 40 cents. How many did he break? *Ans.* 5.

24. A person engaged to work a days on these conditions: For each day he worked he was to receive b cents, for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

Ans. $\frac{ab - d}{b + c}$ days.

25. It is required to divide the number 204 into two such parts, that $\frac{2}{3}$ of the less being taken from the greater, the remainder will be equal to $\frac{3}{4}$ of the greater subtracted from 4 times the less. *Ans.* The numbers are 154 and 50.

26. A gambler staked $\frac{1}{2}$ of his money and lost, but afterward won \$4; he then lost $\frac{1}{2}$ of what he had, and afterward won \$3; after this he lost $\frac{1}{2}$ of what he had, and found he had \$20 remaining. How much had he at first?

Ans. \$30.

27. A man spends $\frac{3}{5}$ of his yearly income for the support of his family, and $\frac{3}{4}$ of the remainder for improving his house and grounds, and lays by \$70 a year. What is his income?

Ans. \$630.

28. My horse and saddle are together worth \$90, and my horse is worth 8 times as much as my saddle. What is the value of each? *Ans.* Saddle, \$10; horse, \$80.

29. My horse and saddle are together worth \$$a$, and my horse is worth n times as much as my saddle. What is the value of each? *Ans.* Saddle, $\$\frac{a}{n+1}$; horse, $\$\frac{na}{n+1}$.

30. The rent of an estate is 8 per cent greater this year than last. This year it is 1890 dollars. What was it last year? *Ans.* \$1750.

31. The rent of an estate is n per cent greater this year than last. This year it is a dollars. What was it last year?

Ans. $\$\frac{100a}{100+n}$.

32. What number is that of which $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{2}{7}$ added together make 73 (a)? *Ans.* 84. General *Ans.* $\frac{84a}{73}$.

33. A and B have the same income. A contracts an annual debt amounting to $\frac{1}{4}$ of it; B lives upon $\frac{4}{5}$ of it; at the end of two years B lends to A enough to pay off his debts, and has 32 (a) dollars to spare. What is the income of each?

Ans. \$280 or $\frac{35a}{4}$.

34. A person after spending 100 dollars more than $\frac{1}{5}$ of his income, had remaining 35 dollars more than $\frac{1}{8}$ of it. Required his income. *Ans.* \$450.

35. A person after spending a dollars more than $\frac{1}{4}$ of his income, had remaining b dollars more than $\frac{1}{5}$ of it. Required his income. *Ans.* $\frac{21\,(a+b)}{11}$ dollars.

36. A sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours afterward B sets out from the same place in pursuit, at the rate of 5 miles in 3 hours. How long and how far must B travel before he overtakes A? *Ans.* 42 hours, and 70 miles.

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

29. Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{l} 3x+2y=118 \\ x+5y=191 \end{array}\right\}$. *Ans.* $x=16,\ y=35$.

2. $\left\{\begin{array}{l} 2x+7y=34 \\ 5x+9y=51 \end{array}\right\}$. *Ans.* $x=3,\ y=4$.

3. $\left\{\begin{array}{l} x+4y=16 \\ 4x+y=34 \end{array}\right\}$. *Ans.* $x=8,\ y=2$.

4. $\left\{\begin{matrix} 12x + 13y - 37 = 0 \\ 17x - 19y - 15 = 0 \end{matrix}\right\}$. *Ans.* $x = 2,\ y = 1.$

5. $\left\{\begin{matrix} 2x - 9y = 11 \\ 3x - 12y = 15 \end{matrix}\right\}$. *Ans.* $x = 1,\ y = -1.$

6. $\left\{\begin{matrix} 9x - 4y = 8 \\ 13x + 7y = 101 \end{matrix}\right\}$. *Ans.* $x = 4,\ y = 7.$

7. $\left\{\begin{matrix} 3y - x = 104 \\ 5x - 2y = 78 \end{matrix}\right\}$. *Ans.* $x = 34,\ y = 46.$

8. $\left\{\begin{matrix} 7y + 7 = 3x \\ 11x + 5y = 87 \end{matrix}\right\}$. *Ans.* $x = 7,\ y = 2.$

9. $\left\{\begin{matrix} 5x = 9 + 3x \\ 5y = 16 - 2x \end{matrix}\right\}$. *Ans.* $x = 3,\ y = 2.$

10. $\left\{\begin{matrix} 2x + 3y = 32 \\ 11y - 9x = 3 \end{matrix}\right\}$. *Ans.* $x = 7,\ y = 6.$

11. $\left\{\begin{matrix} 3y - 4x - 1 = 0 \\ 18 - 3x = 4y \end{matrix}\right\}$. *Ans.* $x = 2,\ y = 3.$

12. $\left\{\begin{matrix} ax - by = 0 \\ x + y = c \end{matrix}\right\}$. *Ans.* $x = \dfrac{bc}{a+b},\ y = \dfrac{ac}{a+b}.$

13. $\left\{\begin{matrix} ax + by = c \\ px + qy = r \end{matrix}\right\}$. *Ans.* $x = \dfrac{cq - br}{aq - bp},\ y = \dfrac{ar - cp}{aq - bp}.$

14. $\left\{\begin{matrix} ax - by = m \\ cx + ey = n \end{matrix}\right\}$. *Ans.* $x = \dfrac{em + bn}{ae + bc},\ y = \dfrac{an - cm}{ae + bc}.$

15. $\left\{\begin{matrix} x + ay = b \\ ax - by = c \end{matrix}\right\}$. *Ans.* $x = \dfrac{ac + b^2}{a^2 + b},\ y = \dfrac{ab - c}{a^2 + b}.$

16. $\left\{\begin{array}{l} ax + 2by = c \\ a(3a - x) = b(x + y) \end{array}\right\}$.

Ans. $x = \dfrac{6a^2 - c}{a + 2b}$, $y = \dfrac{ac + bc - 3a^2}{b(a + 2b)}$.

17. $\left\{\begin{array}{l} ax + by = c \\ bx - ay = d \end{array}\right\}$. *Ans.* $x = \dfrac{ac + bd}{a^2 + b^2}$, $y = \dfrac{bc - ad}{a^2 + b^2}$.

18. $\left\{\begin{array}{l} 7x + 3y = 11 \\ 3xy = (x + 1)(3y + 1) \end{array}\right\}$. *Ans.* $x = 2$, $y = -1$.

19. $\left\{\begin{array}{l} (x + 5)(y + 7) = (x + 1)(y - 9) + 112 \\ 2x + 10 - (3y + 1) = 0 \end{array}\right\}$.

Ans. $x = 3$, $y = 5$.

20. $\left\{\begin{array}{l} \dfrac{x}{3} + 3y = 39 \\ \dfrac{y}{3} + 3x = 31 \end{array}\right\}$. *Ans.* $x = 9$, $y = 12$.

21. $\left\{\begin{array}{l} \dfrac{x}{9} + \dfrac{y}{8} = 43 \\ \dfrac{x}{8} + \dfrac{y}{9} = 42 \end{array}\right\}$. *Ans.* $x = 144$, $y = 216$.

22. $\left\{\begin{array}{l} x + y = 8 \\ \dfrac{x - y}{2} + \dfrac{2x - y}{3} = -\dfrac{4}{3} \end{array}\right\}$. *Ans.* $x = 2\frac{2}{3}$, $y = 5\frac{1}{3}$.

23. $\left\{\begin{array}{l} \dfrac{x}{6} + \dfrac{y}{7} = 5 \\ 2x + y = 50 \end{array}\right\}$. *Ans.* $x = 18$, $y = 14$.

24. $\left\{\begin{array}{l} \dfrac{x}{a} - \dfrac{y}{b} = m \\ \dfrac{x}{c} + \dfrac{y}{d} = n \end{array}\right\}$.

Ans. $x = \dfrac{ac(bm + dn)}{ad + bc}$, $y = \dfrac{bd(cn - am)}{ad + bc}$.

25. $\left\{\begin{array}{l} \frac{x}{2}-\frac{y}{3}-1=0 \\ \frac{2x-1}{2}-\frac{3y-1}{3}=\frac{5}{6} \end{array}\right\}$. *Ans.* $x=4,\ y=3$.

26. $\left\{\begin{array}{l} 5x-8\frac{1}{2}=7y-14 \\ 2x=y+\frac{5}{4} \end{array}\right\}$. *Ans.* $x=1\frac{1}{6},\ y=1\frac{13}{21}$.

27. $\left\{\begin{array}{l} \frac{7+x}{5}-\frac{2x-y}{4}=3y-5 \\ \frac{5y-7}{2}+\frac{4x-3}{6}=18-5x \end{array}\right\}$. *Ans.* $x=3,\ y=2$.

28. $\left\{\begin{array}{l} \frac{x}{7}+\frac{y}{9}=11 \\ \frac{x}{9}+\frac{y}{2}=16 \end{array}\right\}$. *Ans.* $x=63,\ y=18$.

29. $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}=7 \\ \frac{x}{3}-\frac{y}{8}=\frac{1}{2} \end{array}\right\}$. *Ans.* $x=6,\ y=12$.

30. $\left\{\begin{array}{l} \frac{x}{3}+\frac{y}{2}=\frac{4}{3} \\ \frac{x}{2}+\frac{y}{3}=\frac{7}{6} \end{array}\right\}$. *Ans.* $x=1,\ y=2$.

31. $\left\{\begin{array}{l} 8+\frac{1}{3}y=y-2 \\ y+\frac{1}{4}x=x+6 \end{array}\right\}$. *Ans.* $x=4,\ y=9$.

32. $\left\{\begin{array}{l} \frac{x}{4}+\frac{y}{7}=14 \\ 2x-y=7 \end{array}\right\}$. *Ans.* $x=28,\ y=49$.

33. $\left\{\begin{array}{l} 2x - \dfrac{y-3}{5} = 4 \\ 3y = 9 - \dfrac{x-2}{3} \end{array}\right\}$. *Ans.* $x = 2,\ y = 3.$

34. $\left\{\begin{array}{l} \dfrac{x+y}{3} + \dfrac{3x-2y}{4} = \dfrac{3}{4} \\ 17x - 31y = \dfrac{3}{2} \end{array}\right\}$. *Ans.* $x = \dfrac{92}{123},\ y = \dfrac{89}{246}.$

35. $\left\{\begin{array}{l} \dfrac{x+y}{3} + \dfrac{x-y}{4} = 59 \\ \dfrac{x}{11} = \dfrac{3y}{5} \end{array}\right\}$. *Ans.* $x = 99,\ y = 15.$

36. $\left\{\begin{array}{l} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 8 \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11 \end{array}\right\}$. *Ans.* $x = 18,\ y = 6.$

37. $\left\{\begin{array}{l} .1y + .125x = y - x \\ y - .5x = 3y - 3x \end{array}\right\}$. *Ans.* $x = 4,\ y = 5.$

38. $\left\{\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 - \dfrac{x}{c} \\ \dfrac{y}{a} + \dfrac{x}{b} = 1 + \dfrac{y}{c} \end{array}\right\}$.

Ans. $x = \dfrac{(ab + ac - bc)\,abc}{a^2b^2 + a^2c^2 - b^2c^2},\ y = \dfrac{(ac - ab - bc)\,abc}{a^2b^2 + a^2c^2 - b^2c^2}.$

39. $\left\{\begin{array}{r} 24 - (y - x) = 6\,(y - 17\frac{2}{3}) \\ y = x + 10 \end{array}\right\}$. *Ans.* $x = 10,\ y = 20.$

SIMPLE EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

30. Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{l} x + y + z = 53 \\ x + 2y + 3z = 105 \\ x + 3y + 4z = 134 \end{array}\right\}$. *Ans.* $x = 24$, $y = 6$, $z = 23$.

2. $\left\{\begin{array}{l} 2x + 3y + 4z = 29 \\ 3x + 2y + 5z = 32 \\ 4x + 3y + 2z = 25 \end{array}\right\}$. *Ans.* $x = 2$, $y = 3$, $z = 4$.

3. $\left\{\begin{array}{l} \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62 \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47 \\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38 \end{array}\right\}$. *Ans.* $x = 24$, $y = 60$, $z = 120$.

4. $\left\{\begin{array}{l} x + y + z = 90 \\ 2x + 40 = 3y + 20 \\ 2x + 40 = 4z + 10 \end{array}\right\}$. *Ans.* $x = 35$, $y = 30$, $z = 25$.

5. $\left\{\begin{array}{l} 5x + 3y = 65 \\ 2y - z = 11 \\ 3x + 4z = 57 \end{array}\right\}$. *Ans.* $x = 7$, $y = 10$, $z = 9$.

6. $\left\{\begin{array}{l} 12x - \frac{4}{5}y + 16z = 9\frac{11}{15} \\ \frac{x}{3} + 4\frac{1}{2}y + 32z = 9\frac{2}{3} \\ \frac{16}{5}x - 4\frac{1}{2}y + 4z = 1\frac{1}{10} \end{array}\right\}$. *Ans.* $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$.

7. $\left\{\begin{matrix} x+y+z=31 \\ x+y-z=25 \\ x-y-z=9 \end{matrix}\right\}$. *Ans.* $x=20,\ y=8,\ z=3.$

8. $\left\{\begin{matrix} x+y+z=26 \\ x-y\ \ \ =4 \\ x-z\ \ \ =6 \end{matrix}\right\}$. *Ans.* $x=12,\ y=8,\ z=6.$

9. $\left\{\begin{matrix} x-y-z=6 \\ 3y-x-z=12 \\ 7z-y-x=24 \end{matrix}\right\}$. *Ans.* $x=39,\ y=21,\ z=12.$

10. $\left\{\begin{matrix} x+\frac{1}{2}y=100 \\ y+\frac{1}{3}z=100 \\ z+\frac{1}{4}x=100 \end{matrix}\right\}$. *Ans.* $x=64,\ y=72,\ z=84.$

11. $\left\{\begin{matrix} x+y=a \\ x+z=b \\ y+z=c \end{matrix}\right\}$.

Ans. $x=\frac{a+b-c}{2},\ y=\frac{a+c-b}{2},\ z=\frac{b+c-a}{2}.$

12. $\left\{\begin{matrix} u+v+x+y=10 \\ u+v+x+z=11 \\ u+v+y+z=12 \\ u+x+y+z=13 \\ v+x+y+z=14 \end{matrix}\right\}$

Ans. $x=3,\ y=4,\ z=5,\ u=1,\ v=2.$

13. $\left\{\begin{matrix} 2x=u+y+z \\ 3y=u+x+z \\ 4z=u+x+y \\ u=x-14 \end{matrix}\right\}$.

Ans. $u=26,\ x=40,\ y=30,\ z=24.$

14. $\left.\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=58 \end{array}\right\}$. *Ans.* $x=24$, $y=60$, $z=120$.

15. $\left.\begin{array}{l} x+a=\ y+z \\ y+a=2x+\ 2z \\ z+a=3x+3y \end{array}\right\}$.

Ans. $x=\frac{1}{11}a$, $y=\frac{5}{11}a$, $z=\frac{7}{11}a$.

16. $\left.\begin{array}{l} 2x+y-2z=40 \\ 4y-x+3z=35 \\ 3u+t=13 \\ \ \ y+u+t=15 \\ 3x-y+3t-u=49 \end{array}\right\}$.

Ans. $x=20$, $y=10$, $z=5$, $u=4$, $t=1$.

PROBLEMS.

31.—1. What two numbers are those, to one-third the sum of which if I add 13, the result will be 17; and if from half their difference I subtract 1, the result will be 2?

Ans. 9 and 3.

2. Find a fraction, such that if I add 1 to its numerator, it becomes $\frac{1}{2}$, and if I add 3 to the denominator, it becomes $\frac{1}{3}$. *Ans.* $\frac{5}{13}$.

3. A rectangular room having been measured, it was observed that if it were 5 feet broader and 4 feet longer it would contain 116 square feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Find its length and breadth.

Ans. Length, 12 feet; breadth, 9 feet.

4. A cask B contains 12 gallons of wine and 4 gallons of water; another cask C contains 8 gallons of wine and 12 gallons of water; how many gallons must be drawn from each cask so as to produce, by their mixture, 7 gallons of wine and 7 gallons of water?

Ans. 4 gallons from B, and 10 gallons from C.

5. A sum of money was to be divided among three persons, A, B, and C, so that A's share should exceed $\frac{1}{4}$ of the sum of the shares of B and C by \$120, the share of B should exceed $\frac{1}{5}$ of the sum of the shares of A and C by \$120, and the share of C, likewise, should exceed $\frac{1}{6}$ of the sum of the shares of A and B by \$120. What was the share of each?

Ans. A's share, \$600; B's, \$480; C's, \$360.

6. A and B together can earn \$40 in 6 days; A and C together can earn \$54 in 9 days; and B and C together can earn \$80 in 15 days. What can each person earn in a day?

Ans. A, \$3$\frac{2}{3}$; B, \$3; C, \$2$\frac{1}{3}$.

7. The sum of the ages of A, B, and C is 18 years; the sum of the ages of A, B, and D is 16 years; the sum of the ages of A, C, and D is 14 years; and the sum of the ages of B, C, and D is 12 years. What is the age of each?

Ans. A, 8 years; B, 6 years; C, 4 years; D, 2 years.

8. A man left a sum of money to be divided among four servants, so that the share of the first should be $\frac{1}{2}$ the sum of the shares of the three others; the share of the second, $\frac{1}{3}$ of the sum of the shares of the three others; the share of the third, $\frac{1}{4}$ of the sum of the shares of the three others; and it was found that the share of the last was \$14 less than that of the first. What was the amount of money divided, and what was the share of each servant?

Ans. Amount divided, \$120; shares, \$40, \$30, \$24, \$26.

9. A jockey has two horses and two saddles, one of the saddles being worth \$15 and the other \$10. Now, if the better saddle be put on the better horse, the value of the better horse and saddle will be $\frac{4}{5}$ of the value of the other horse and saddle; but if the better saddle be put on the poorer horse, and the poorer saddle on the better horse, the value of the better horse and saddle will be $1\frac{2}{13}$ that of the other horse and saddle. What is the value of each horse?

Ans. Better, \$65; other, \$50.

10. A and B together can perform a piece of work in 16 days. They work together for four days, when A being called off, B is left to finish the work, which he does in 36 days. In what time could each alone do the work?

Ans. A in 24 days; B in 48 days.

11. If the numerator of a certain fraction be doubled, and the denominator be increased by 7, the resulting fraction is $\frac{2}{3}$; but if the denominator be doubled and the numerator increased by 2, the result is $\frac{3}{5}$. Find the fraction. *Ans.* $\frac{4}{7}$.

12. Some hours after a courier had started from A to B, which are 147 miles distant from each other, a second courier was sent, who wished to reach B at the same time with the first; to accomplish this, he must travel the distance from A to B in 28 hours less time than the first did. Now, the time in which the first travels 17 miles added to the time in which the second travels 56 miles is $13\frac{1}{3}$ hours. How many miles does each travel per hour?

Ans. 1st, 3; the 2d, 7 miles per hour.

13. Find two numbers such that the sum of $\frac{1}{2}$ the greater and $\frac{1}{3}$ the less shall be 13, and the difference between $\frac{1}{2}$ the greater and $\frac{1}{3}$ the less shall be nothing. *Ans.* 18 and 12.

14. A man has to travel a certain distance; when he has traveled 20 miles he increases his speed 1 mile per hour; if he had traveled with his increased speed during the whole of his journey, he would have arrived 40 minutes earlier; but if he had kept on at his first rate, he would have arrived 20 minutes later; how far did he travel?

Ans. 30 miles.

15. Find three numbers, such that the first with $\frac{1}{2}$ the sum of the second and third shall be 120; the second with $\frac{1}{3}$ of the difference between the third and first shall be 70; and $\frac{1}{4}$ the sum of the three numbers shall be 95.

Ans. 50, 65, 75.

16. A number is expressed by three digits; the sum of these digits is 11; the figure in the units' place is double that in the hundreds' place; and when 297 is added to the number, the sum obtained is expressed by the same digits in an inverted order. What is the number? *Ans.* 326.

17. Divide the number 90 into three such parts, that twice the first part increased by 40, three times the second part increased by 20, and four times the third part increased by 10, may be equal to each other. *Ans.* 35, 30, 25.

18. If B's age be subtracted from A's, the difference will be C's age; if five times B's age be added to twice C's age, and from the sum A's age be subtracted, the remainder will be 147 years; the sum of all their ages is 96 years. What is the age of each?

Ans. A's, 48 years; B's, 33 years; C's, 15 years.

INTERPRETATION OF NEGATIVE RESULTS.

PROBLEMS.

32.—1. The joint property of A and B amounts to \$120, and A is worth \$160 more than B. What amount of property has each? *Ans.* A, \$140; B, \$—20.

Interpret the negative result.

2. A and B commenced trade at the same time; A had three times as much money as B. A gained \$400 and B \$150, when it appeared that A had twice as much money as B. How much did each have at first?

Ans. A, \$—300; B, \$—100.

Interpret the result, and modify the enunciation so that the result shall be \$300 and \$100.

3. Find a number the one-fourth of which exceeds the one-third of it by 12. *Ans.* —144.

Modify the enunciation so that the result shall be +144.

4. A is 30 years old and B is 15 years old. When will the age of A be three times that of B? *Ans.* $-7\frac{1}{2}$ years.

Interpret this result, and modify the enunciation so that the result shall be $+7\frac{1}{2}$ years.

5. A worked 7 days and B 3 days, and together they received for wages \$22. If A had worked 5 days and B 1 day, they would have received \$18. What were the daily wages of each? *Ans.* A, \$4; B, \$—2.

Interpret this result, and modify the enunciation so that the result shall be \$4 and \$2.

6. A man worked for a person ten days, having his wife with him 8 days, and his son 6 days, and received \$10.30 as compensation for all three; at another time he worked 12 days, his wife 10 days, and son 4 days, and received \$13.20; at another time he worked 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and received \$13.85. What were the daily wages of each?

Ans. Man, \$0.75; wife, \$.50; son, \$—.20.

Interpret this result, and modify the enunciation so that the result shall be \$.75, \$.50, \$.20.

7. B is d miles in advance of A; B travels b miles per hour, and A travels a miles per hour. When will A overtake B?

Ans. In $\frac{d}{a-b}$ hours.

Interpret this result when $a < b$.

VANISHING FRACTIONS.

33.—1. Find the value of $\left(\frac{x-x^5}{1-x}\right)_{x=1}$. *Ans.* 4.

2. Find the value of $\left\{\frac{x^4-1}{(1+x)(1-x)}\right\}_{x=1}$.

Ans. -2.

3. Find the value of $\left\{\frac{a^2-b^2}{(a-b)^2}\right\}_{a=b}$. *Ans.* ∞.

4. Find the value of $\left(\frac{x^4-2ax^3+2a^3x-a^4}{x^3-ax}\right)_{x=a}$.

Ans. 0.

5. Find the value of $\left\{\frac{a-x}{a+x}\times\frac{(a+x)^2}{a^2-x^2}\right\}_{x=a}$.

Ans. 1.

6. Find the value of $\left\{\frac{\frac{1-x}{x-x^5}}{\frac{(a-b)^2}{a^2-b^2}}\right\}_{\substack{x=1,\\ a=b.}}$ *Ans.* ∞.

INDETERMINATE EQUATIONS.

34.—1. Is the equation $x+y=10$ indeterminate? Why?

2. Are the equations

$$\left\{\begin{matrix} x+y+z=100 \\ x-y-z=\ 50 \end{matrix}\right\}$$

indeterminate? Why?

3. What relation must subsist between m, n, p, and q in order that the equation $mx-n=px-q$ may be indeterminate? *Ans.* $m=p$ and $n=q$.

4. What relation must subsist between a, m, n, p, r, and s in order that the equations

$$\left\{\begin{matrix} ax-my=r \\ nx-py=s \end{matrix}\right\}$$

may be indeterminate? *Ans.* $pr=ms$, $ap=mn$.

5. How many values have x and y in the equations

$$\left\{\begin{matrix} 8x-4y=2 \\ 12x-6y=3 \end{matrix}\right\}?$$

Ans. An infinite number.

INDETERMINATE PROBLEMS.

35.—1. Find two positive integers whose sum is 5.

Ans. 1, 4; 2, 3.

2. Find two positive integers, such that if the first be added to twice the second, the sum will be 12.

Ans. 10, 1; 8, 2; 6, 3; 4, 4; 2, 5.

3. Find two positive integers, such that if four times the first be added to three times the second, the sum will be 24.

Ans. 3 and 4.

4. A man paid $500 for cows and sheep; he bought cows at $17 each, and sheep at $5 each. How many of each did he buy?

Ans. 5 cows and 83 sheep; 10 cows and 66 sheep; 15 cows and 49 sheep; 20 cows and 32 sheep; or 25 cows and 15 sheep.

5. A boy had between two and three dozen apples; when he counted them out by 4 at a time, there were three left; and when he counted them out by 5 at a time there was one left. How many apples had he? *Ans.* 31.

6. Find a positive integer, such that if it be divided by 5, 6, and 7, the corresponding remainders may be 4, 5, and 6.

Ans. 209.

7. A boy had between one and two dozen oranges; when he counted them out by 2 at a time, there was one left; when he counted them out by 3 at a time, there were two left; and when he counted them out by 4 at a time, there were three left. How many oranges had he? *Ans.* 23.

INCOMPATIBLE EQUATIONS.

36.—1. Are the equations

$$\left\{\begin{array}{l} x+y=5 \\ x-y=1 \\ \dfrac{x}{y}=6 \end{array}\right\}$$

incompatible? Why?

2. Find the values of x and y in the equations

$$\left\{\begin{array}{l} x+\ y=8 \\ 2x+3y=1 \\ x=2y \end{array}\right\}.$$

Ans. The equations are incompatible.

3. Find the relation which must subsist among the quantities a, b, c, d, e, f, and s, in order that the equations

$$\left\{\begin{array}{l} ax+\ by=s \\ cx-\ dy=e \\ x=fy \end{array}\right\}$$

may be compatible. *Ans.* $be+ds=f(cs-ae)$.

4. How many conditions are there in the problem which would furnish the equations

$$\left\{\begin{array}{l} x+y=10 \\ x-y=\ 2 \\ x=\ 8y \end{array}\right\}?$$

Is the problem possible? Why?

INEQUALITIES.

REDUCTION OF INEQUALITIES.

37. Reduce the following inequalities:

1. $7x - \frac{23}{3} > \frac{2x}{3} + 5.$ *Ans.* $x > 2.$

2. $\frac{5x+7}{2} > \frac{3x-8}{5}.$ *Ans.* $x > -\frac{51}{19}.$

3. $3x - \frac{x+2}{4} < \frac{2x+3}{5}.$ *Ans.* $x < \frac{22}{47}.$

4. $\frac{\frac{3x}{2} + \frac{4x}{3}}{x - \frac{x-2}{3}} < \frac{x - \frac{x}{4}}{x + \frac{x}{4}}.$ *Ans.* $x < \frac{6}{79}.$

5. $\frac{2x+a}{b} - \frac{x-b}{a} > \frac{3ax + (a-b)^2}{ab}.$

Ans. $x > \frac{2ab}{a+b}.$

6. $\frac{a}{x} + \frac{b}{a} - \frac{a}{3ax} > 0.$ *Ans.* $x > \frac{a(1-3a)}{3b}.$

COMBINATION OF AN EQUATION WITH AN INEQUALITY.

38. Find a limit for x and y in the following groups:

1. $\left\{\begin{matrix} 4x + 3y > 31 \\ 3x + 2y = 22 \end{matrix}\right\}.$ *Ans.* $x < 4,\ y > 5.$

2. $\left\{\begin{matrix} 5x - 4y < 19 \\ 4x + 2y = 36 \end{matrix}\right\}.$ *Ans.* $x < 7,\ y > 4.$

3. $\left\{\begin{array}{l} \dfrac{x+y}{3} - 2y > 2 \\ \dfrac{2x-4y}{5} + y = \dfrac{23}{5} \end{array}\right\}$. *Ans.* $x > 11$, $y < 1$.

4. $\left\{\begin{array}{l} \dfrac{x}{3y} < 1\frac{1}{3} \\ \dfrac{2x+7y}{5} - 1 = \dfrac{2(2x-6y+1)}{3} \end{array}\right\}$.

Ans. $x < 4$, $y < 1$.

5. $\left\{\begin{array}{l} \dfrac{x+1}{y} > \dfrac{1}{2} \\ \dfrac{x}{y+3} = \dfrac{1}{3} \end{array}\right\}$. *Ans.* $x > 5$, $y > 12$.

6. $\left\{\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} < 1 + \dfrac{x}{4} \\ \dfrac{y}{2} + \dfrac{x}{3} = 1 + \dfrac{y}{4} \end{array}\right\}$. *Ans.* $x > 1\frac{5}{7}$, $y < 1\frac{5}{7}$.

INVOLUTION AND EVOLUTION.

INVOLUTION.

POWERS OF MONOMIALS.

39. Verify the following equations:

1. $(a^9)^3 = a^{27}$.
2. $(-a^7b^8)^5 = -a^{35}b^{40}$.
3. $(a^2)^5 = a^{10}$.
4. $(-a^3)^3 = -a^9$.
5. $(ab^3)^2 = a^2b^6$.
6. $(3a^2x^2z^3)^5 = 243a^{10}x^{10}z^{15}$.
7. $(-a^n)^m = a^{mn}$ if m is even.
8. $(-a^n)^m = -a^{mn}$ if m is odd.

POWERS OF POLYNOMIALS.

40. Verify the following equations:

1. $(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

2. $(a + b + c - d)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 - 2ad - 2bd - 2cd + d^2$.

3. $(a^3 - 3a^2b + 3ab^2 - b^3)^2 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

4. $(1 - 2x + 3x^2)^3 = 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

5. $(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2$.

6. $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$.

7. $(x - 2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$.

8. $(x + 3)^5 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$.

9. $(2m - 1)^3 = 8m^3 - 12m^2 + 6m - 1$.

10. $(3x + 1)^4 = 81x^4 + 108x^3 + 54x^2 + 12x + 1$.

11. $(2x - a)^4 = 16x^4 - 32ax^3 + 24a^2x^2 - 8a^3x + a^4$.

12. $(1 + 2x - 3x^2)^5 = 1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$.

POWERS OF FRACTIONS.

41. Verify the following equations:

1. $\left(\frac{a^2}{b^3}\right)^8 = \frac{a^{16}}{b^{24}}$.

2. $\left(-\frac{x^2}{y^5}\right)^4 = \frac{x^8}{y^{20}}$.

3. $\left(-\frac{a^2x^3y^4}{b^5c^6z^7}\right)^5 = -\frac{a^{10}x^{15}y^{20}}{b^{25}c^{30}z^{35}}$.

4. $\left(\frac{a+b}{a-b}\right)^2 = \frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

5. $\left(\frac{a-b}{a+b}\right)^3 = \frac{a^3-3a^2b+3ab^2+b^3}{a^3+3a^2b+3ab^2+b^3}$.

6. $\left(\frac{x}{a}-\frac{a}{x}\right)^3 = \frac{x^3}{a^3}-\frac{3x}{a}+\frac{3a}{x}-\frac{a^3}{x^3}$.

EVOLUTION.

ROOTS OF MONOMIALS.

42. Verify the following equations by extracting the indicated roots:

1. $\sqrt{4x^2y^2} = \pm 2xy$.
2. $\sqrt[3]{a^3b^3} = ab$.
3. $\sqrt{x^4y^2} = \pm x^2y$.
4. $\sqrt{16a^2b^2c^4} = \pm 4abc^2$.
5. $\sqrt{81a^4b^4} = \pm 9a^2b^2$.
6. $\sqrt[3]{a^3b^6c^9} = ab^2c^3$.
7. $\sqrt[3]{-64a^3x^6} = -4ax^2$.
8. $\sqrt[3]{x^{3m}y^{6n}} = x^my^{2n}$.

SQUARE ROOT OF POLYNOMIALS.

43. Verify the following equations by extracting the indicated roots:

1. $\sqrt{(a^2-4ab+4b^2)} = \pm(a-2b)$.
2. $\sqrt{(a^2-2ab+c^2+b^2+2ac-2bc)} = \pm(a-b+c)$.
3. $\sqrt{(x^6+4x^5+2x^4+9x^2-4x+4)} = \pm(x^3+2x^2-x+2)$.

4. $\sqrt{(x^4 + 2ax^3 + 3a^2x^2 + 2a^3x + a^4)} = \pm (x^2 + ax + a^2)$.

5. $\sqrt{(x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4)} = \pm (x^2 - 3xy + 2y^2)$.

6. $\sqrt{(1 - x^2 + 25x^4 - 30x^3 + 6x)} = \pm (1 + 3x - 5x^2)$.

7. $\sqrt{(4x^4 + 12ax^3 + 13a^2x^2 + 6a^3x + a^4)} = \pm (2x^2 + 3ax + a^2)$.

8. $\sqrt{(x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8)} = \pm (x^4 - a^2x^2 - a^4)$.

9. $\sqrt{(4a^2 + 9b^2 + 16c^2 - 12ab + 16ac - 24bc)} = \pm (2a - 3b + 4c)$.

10. $\sqrt{(9a^2 - 12ab + 30ac - 6ad + 4b^2 - 20bc + 4bd + 25c^2 - 10cd + d^2)} = \pm (3a - 2b + 5c - d)$.

CUBE ROOT OF POLYNOMIALS.

44. Verify the following equations by extracting the indicated roots:

1. $\sqrt[3]{(a^3 + 12a^2 + 48a + 64)} = a + 4$.

2. $\sqrt[3]{(a^3 - 3a^2x + 3ax^2 - x^3)} = a - x$.

3. $\sqrt[3]{(8x^3 + 12x^2 + 6x + 1)} = 2x + 1$.

4. $\sqrt[3]{(27a^3 - 54a^2x + 36ax^2 - 8x^3)} = 3a - 2x$.

5. $\sqrt[3]{(27x^3 - 135x^2 + 225x - 125)} = 3x - 5$.

6. $\sqrt[3]{(a^3b^6 - 12a^2b^4 + 48ab^2 - 64)} = ab^2 - 4$.

7. $\sqrt[3]{(125a^6c^3 - 225a^4c^2xy^2 + 135a^2cx^2y^4 - 27x^3y^6)} = 5a^2c - 3xy^2$.

8. $\sqrt[3]{8x^6 - 36x^4y^2 + 54x^2y^4 - 27y^6)} = 2x^2 - 3y^2$.

9. $\sqrt[3]{a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1)} = a^2 - 2a + 1$.

10. $\sqrt[3]{(a^{3m} - 6a^{2m+1}x^n + 12a^{m+2}x^{2n} - 8a^3x^{3n})} = a^m - 2ax^n$.

HIGHER ROOTS.

45. Verify the following equations by extracting the indicated roots:

1. $\sqrt[4]{(a^4 + 8a^3 + 24a^2 + 32a + 16)} = \pm (a + 2)$.

2. $\sqrt[4]{(x^4 - 8x^3 + 24x^2 - 32x + 16)} = \pm (x - 2)$.

3. $\sqrt[4]{(81x^4 + 108x^3 + 54x^2 + 12x + 1)} = \pm (3x + 1)$.

4. $\sqrt[6]{(x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)} = \pm (x - 1)$.

5. $\sqrt[8]{(1 - 8x^2 + 28x^4 - 56x^6 + 70x^8 - 56x^{10} + 28x^{12} - 8x^{14} + x^{16})} = \pm (1 - x^2)$.

6. $\sqrt[9]{(a^{9m} + 9a^{8m}x^n + 36a^{7m}x^{2n} + 84a^{6m}x^{3n} + 126a^{5m}x^{4n} + 126a^{4m}x^{5n} + 84a^{3m}x^{6n} + 36a^{2m}x^{7n} + 9a^m x^{8n} + x^{9n})} = a^m + x^n$.

ROOTS OF FRACTIONS.

46. Verify the following equations by extracting the indicated roots:

1. $\sqrt{\frac{a^4}{b^4}} = \pm \frac{a^2}{b^2}$.

2. $\sqrt[4]{\frac{16a^8b^8}{c^4}} = \pm \frac{2a^2b^2}{c^2}$.

3. $\sqrt{\frac{2^{16}b^{2x}}{9}} = \pm \frac{2^8 b^x}{3}$.

4. $\sqrt[3]{\frac{125a^6b^{12}}{x^3y^{15}}} = \frac{5a^2b^4}{xy^5}$.

5. $\sqrt{\frac{a^2 - 4ab + 4b^2}{x^6 + 4x^5 + 2x^4 + 9x^2 - 4x + 4}} = \pm \frac{a - 2b}{x^3 + 2x^2 - x + 2}$.

6. $\sqrt[4]{\dfrac{x^4-8x^3+24x^2-32x+16}{81x^4+108x^3+54x^2+12x+1}}=\pm\dfrac{x-2}{3x+1}$.

7. $\sqrt{\dfrac{x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}}{x^2+\frac{2ax}{3}-bx+\frac{a^2}{9}+\frac{b^2}{4}-\frac{ab}{3}}}=\pm\dfrac{x^2-x+\frac{1}{4}}{x+\frac{a}{3}-\frac{b}{2}}$.

THEORY OF EXPONENTS.

47.—1. Find the cube root of $a^{\frac{1}{2}}b^{\frac{3}{4}}c^3$. *Ans.* $a^{\frac{1}{6}}b^{\frac{1}{4}}c$.

2. Find the fourth root of $a^{-8}b^{-2}c^{\frac{1}{4}}$. *Ans.* $a^{-2}b^{-\frac{1}{2}}c^{\frac{1}{16}}$.

3. Find the value of $\sqrt[3]{a^{3m}b^{-n}c^{\frac{1}{2n}}}$. *Ans.* $a^m b^{-\frac{n}{3}}c^{\frac{1}{6n}}$.

4. Find the fourth power of $a^{\frac{1}{2}}-x^{\frac{1}{2}}$.

Ans. $a^2-4a^{\frac{3}{2}}x^{\frac{1}{2}}+6ax-4a^{\frac{1}{2}}x^{\frac{3}{2}}+x^2$.

5. Find the fifth power of $a^{\frac{1}{3}}+x^{\frac{1}{3}}$.

Ans. $a^{\frac{5}{3}}+5a^{\frac{4}{3}}x^{\frac{1}{3}}+10ax^{\frac{2}{3}}+10a^{\frac{2}{3}}x+5a^{\frac{1}{3}}x^{\frac{4}{3}}+x^{\frac{5}{3}}$.

6. Find the fifth power of $a^{-1}-y^{-1}$.

Ans. $a^{-5}-5a^{-4}y^{-1}+10a^{-3}y^{-2}-10a^{-2}y^{-3}+5a^{-1}y^{-4}-y^{-5}$.

7. Multiply $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. *Ans.* $x-y$.

8. Multiply $2x^{\frac{3}{4}}-y^{\frac{5}{6}}$ by $2x^{\frac{3}{4}}+y^{\frac{5}{6}}$. *Ans.* $4x^{\frac{3}{2}}-y^{\frac{5}{3}}$.

9. Multiply $\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{1}{3}}+\frac{1}{4}x^{\frac{1}{4}}$ by $\frac{1}{4}x^{\frac{1}{4}}+\frac{1}{3}x^{\frac{1}{3}}+\frac{1}{2}x^{\frac{1}{2}}$.

Ans. $\frac{1}{4}x-\frac{1}{9}x^{\frac{2}{3}}+\frac{1}{4}x^{\frac{3}{4}}+\frac{1}{16}x^{\frac{2}{3}}$.

10. Divide $x^2y^{\frac{2}{3}}z^3$ by $-2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$. *Ans.* $-\frac{1}{2}x^{\frac{3}{2}}y^{\frac{4}{15}}z^{\frac{11}{4}}$.

11. Divide $a^2b^{\frac{1}{2}} - a^{\frac{3}{2}}b^{\frac{3}{2}} + ab$ by $a^{\frac{1}{2}}b^{\frac{1}{2}}$.

Ans. $a^{\frac{3}{2}}b^{-1} - a^{\frac{1}{2}}b^{-\frac{1}{2}} + a^{\frac{1}{2}}b^{-\frac{1}{2}}$.

12. Divide $x^{\frac{5}{2}} - 4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 6x - x^2$ by $x^{\frac{3}{2}} + 2 - 4x^{\frac{1}{2}}$.

Ans. $x - x^{\frac{1}{2}}$.

13. Divide $\frac{1}{5}x^{\frac{6}{5}} - \frac{3}{10}x^{\frac{7}{10}} + \frac{1}{2}x^{\frac{3}{4}} - \frac{1}{3}x^{\frac{7}{12}} - \frac{3}{20}x^{\frac{3}{10}} + \frac{1}{4}x^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - \frac{3}{5}x^{\frac{1}{5}}$. *Ans.* $\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{1}{4}}$.

14. Find the square root of $x^2 + x^{\frac{3}{2}} - \frac{13}{12}x - \frac{2}{3}x^{\frac{1}{2}} + \frac{4}{9}$.

Ans. $x + \frac{x^{\frac{1}{2}}}{2} - \frac{2}{3}$.

15. Find the cube root of $a^{3x} - 3a^x + 3a^{-x} - a^{-3x}$.

Ans. $a^x - a^{-x}$.

16. Find the cube root of $x^{\frac{3}{2}} - 3x^{\frac{4}{3}} + 3x^{\frac{7}{6}} + 2x + 3x^{\frac{5}{6}} - 3x^{\frac{5}{6}} - 6x^{\frac{13}{12}} + 3x^{\frac{11}{12}} + x^{\frac{3}{4}}$. *Ans.* $x^{\frac{1}{2}} - x^{\frac{1}{3}} + x^{\frac{1}{4}}$.

17. Find the G. C. D. of $1 + x^{\frac{1}{2}} + x + x^{\frac{3}{2}}$ and $2x + 2x^{\frac{3}{2}} + 3x^2 + 3x^{\frac{5}{2}}$ *Ans.* $1 + x^{\frac{1}{2}}$.

18. Find the L. C. M. of $(1+x)^{\frac{3}{2}}$, $(1+x)^{\frac{5}{3}}$, and $(1+x)^{\frac{3}{4}}$.

Ans. $(1+x)^{\frac{6}{3}}$.

RADICAL QUANTITIES.

REDUCTION OF SIMPLE RADICAL QUANTITIES.

48.—1. Reduce abc to a radical quantity of the second degree. *Ans.* $\sqrt{a^2b^2c^2}$.

2. Reduce $(a+b)^2$ to a radical quantity of the third degree. *Ans.* $\sqrt[3]{(a+b)^6}$.

3. Reduce $\frac{m^2}{n^3}$ to a radical quantity of the fourth degree. *Ans.* $\sqrt[4]{\frac{m^8}{n^{12}}}$.

4. Reduce $(a+b)(a-b)$ to a radical quantity of the fifth degree. *Ans.* $\sqrt[5]{(a^2-b^2)^5}$.

5. Reduce $-(m+n)$ to a radical quantity of the eighth degree. *Ans.* $-\sqrt[8]{(m+n)^8}$.

6. Reduce $(m+n)^3$ to a radical quantity of the n^{th} degree. *Ans.* $\sqrt[n]{(m+n)^{3n}}$.

49. Introduce the coefficient of the radical factor, in each of the following expressions, under the radical sign:

1. $5\sqrt{2}$. *Ans.* $\sqrt{50}$.
2. $2\sqrt[3]{3}$. *Ans.* $\sqrt[3]{24}$.
3. $a\sqrt[3]{b}$. *Ans.* $\sqrt[3]{a^3b}$.
4. $a\sqrt[4]{\frac{b}{a}}$. *Ans.* $\sqrt[4]{a^3b}$.

5. $(a-b)\sqrt{\dfrac{a+b}{a^2-2ab+b^2}}$. *Ans.* $\sqrt{a+b}$.

6. $(m+n)\sqrt[n]{bc}$. *Ans.* $\sqrt[n]{bc\,(m+n)^n}$.

7. $ax\sqrt[n]{a^2x^{-3}}$. *Ans.* $\sqrt[n]{a^{n+2}x^{n-3}}$.

8. $(a+b)\,(a+b)^{\frac{1}{3}}$. *Ans.* $[(a+b)^4]^{\frac{1}{3}}$.

9. $a^{-3}(b^{-2}c^{\frac{1}{4}})^{\frac{1}{4}}$. *Ans.* $(a^{-6}b^{-2}c^{\frac{1}{4}})^{\frac{1}{4}}$.

10. $a^{-m}\left(b^{-n}c^{\frac{1}{n}}\right)^{\frac{1}{m}}$. *Ans.* $\left(a^{-mn}b^{-n}c^{\frac{1}{n}}\right)^{\frac{1}{m}}$.

50.—1. Reduce $\sqrt{45}$ to such a form that the factor 9 shall not occur under the radical sign. *Ans.* $3\sqrt{5}$.

2. Reduce $\sqrt[3]{192}$ to such a form that the factor 64 shall not occur under the radical sign. *Ans.* $4\sqrt[3]{3}$.

3. Reduce $4\sqrt{150}$ to such a form that the factor 25 shall not occur under the radical sign. *Ans.* $20\sqrt{6}$.

4. Reduce $(a+b)\sqrt{(a-b)^2c}$ to such a form that the factor $(a-b)^2$ shall not occur under the radical sign.

Ans. $(a^2-b^2)\sqrt{c}$.

5. Reduce $\dfrac{1}{m+n}\sqrt{(m+n)^2x}$ to such a form that the factor $(m+n)^2$ shall not occur under the radical sign.

Ans. $\sqrt{x}$.

6. Reduce $[(a+b)^4]^{\frac{1}{3}}$ to such a form that the factor $(a+b)^3$ shall not occur under the radical sign.

Ans. $(a+b)\,(a+b)^{\frac{1}{3}}$.

51. Reduce each of the following expressions to another in which the quantity under the radical sign shall be entire:

1. $2\sqrt{\frac{1}{3}}$. *Ans.* $\frac{2}{3}\sqrt{3}$.

2. $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$. *Ans.* $\frac{1}{8}\sqrt[3]{180}$.

3. $5\sqrt[3]{\frac{3}{125}}$. *Ans.* $\sqrt[3]{3}$.

4. $a\sqrt{\frac{b}{c}}$. *Ans.* $\frac{a}{c}\sqrt{bc}$.

5. $a\sqrt[n]{\frac{b}{c}}$. *Ans.* $\frac{a}{c}\sqrt[n]{bc^{n-1}}$.

6. $m(a+b)^2\sqrt[n]{\frac{a-b}{a+b}}$.
Ans. $m(a+b)\sqrt[n]{(a-b)(a+b)^{n-1}}$.

7. $(a+b)\sqrt{\frac{a-b}{a+b}}$. *Ans.* $\sqrt{a^2-b^2}$.

8. $(m-n)\sqrt{\frac{m^2c+2cmn+cn^2}{m+n}}$.
Ans. $(m-n)\sqrt{c(m+n)}$.

9. $3\sqrt{\frac{ab^2}{9(a+b)}}$. *Ans.* $\frac{1}{a+b}\sqrt{ab^2(a+b)}$.

10. $\left(\frac{a-b}{a+b}\right)^{\frac{1}{n}}$. *Ans.* $\frac{1}{a+b}[(a^2-b^2)(a+b)^{n-2}]^{\frac{1}{n}}$.

52. Reduce each of the following radical quantities to its simplest form:

1. $\sqrt{75}$. *Ans.* $5\sqrt{3}$.
2. $\sqrt{98a^2}$. *Ans.* $7a\sqrt{2}$.
3. $\sqrt{12x^2y}$. *Ans.* $2x\sqrt{3y}$.
4. $\sqrt[3]{54x^4}$. *Ans.* $3x\sqrt[3]{2x}$.
5. $4\sqrt[3]{108x}$. *Ans.* $12\sqrt[3]{4x}$.
6. $\sqrt{x^3 - a^2x^2}$. *Ans.* $x\sqrt{x - a^2}$.
7. $\sqrt[3]{32a^3}$. *Ans.* $2a\sqrt[3]{4}$.
8. $\sqrt{28a^3x^2}$. *Ans.* $2ax\sqrt{7a}$.
9. $\sqrt{\frac{44}{75}}$. *Ans.* $\frac{2}{15}\sqrt{33}$.
10. $\sqrt[3]{\frac{135}{32}}$. *Ans.* $\frac{3}{4}\sqrt[3]{10}$.
11. $\sqrt[3]{\frac{25}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{75}$.
12. $\sqrt{\frac{50}{147}}$. *Ans.* $\frac{5}{21}\sqrt{6}$.
13. $\sqrt{\frac{ab^2}{4(a+x)}}$. *Ans.* $\frac{b}{2(a+x)}\sqrt{a(a+x)}$.
14. $\sqrt[3]{a^3 + a^3b^3}$. *Ans.* $a\sqrt[3]{1 + b^3}$.
15. $\sqrt[3]{ax^3 + bx^6}$. *Ans.* $x\sqrt[3]{a + bx^3}$.
16. $\sqrt[3]{5(a^3 + a^4b)}$. *Ans.* $a\sqrt[3]{5(1 + ab)}$.

53.—1. Reduce $\sqrt[4]{4a^2}$ to a radical quantity of the second degree. *Ans.* $\sqrt{2a}$.

2. Reduce $\sqrt[6]{8a^3b^3}$ to a radical quantity of the second degree. *Ans.* $\sqrt{2ab}$.

3. Reduce $\sqrt[6]{16a^2b^4}$ to a radical quantity of the third degree. *Ans.* $\sqrt[3]{4ab^2}$.

4. Reduce $\sqrt[8]{25a^4b^6c^8}$ to a radical quantity of the fourth degree. *Ans.* $\sqrt[4]{5a^2b^3c^4}$.

5. Reduce $\sqrt[9]{(a+b)^{12}(c-d)^6}$ to a radical quantity of the third degree. *Ans.* $\sqrt[3]{(a+b)^4(c-d)^2}$.

6. Reduce $\sqrt[10]{a^5(b+c)^{10}d^{15}}$ to a radical quantity of the second degree. *Ans.* $\sqrt{a(b+c)^2d^3}$.

7. Reduce $\sqrt[mn]{a^{2n}b^{3n}c^{mn}}$ to a radical quantity of the m^{th} degree. *Ans.* $\sqrt[m]{a^2b^3c^m}$.

8. Reduce $[(a-b)^m(c-d)^{2mn}]^{\frac{1}{mn}}$ to a radical quantity of the n^{th} degree. *Ans.* $[(a-b)(c-d)^{2n}]^{\frac{1}{n}}$.

54.—1. Reduce $\sqrt{5}$ to a radical quantity of the sixth degree. *Ans.* $\sqrt[6]{125}$.

2. Reduce $\sqrt{a^2b^3}$ to a radical quantity of the twelfth degree. *Ans.* $\sqrt[12]{a^{12}b^{18}}$.

3. Reduce $\sqrt[3]{(a+b)^4(a-b)^4}$ to a radical quantity of the sixth degree. *Ans.* $\sqrt[6]{(a^2-b^2)^8}$.

4. Reduce $\sqrt[10]{a^4b^8c^{16}}$ to a radical quantity of the fifth degree.

Ans. $\sqrt[5]{ab^2c^4}$.

5. Reduce $\sqrt[2mn]{a^{mn}b^{2mn}c^{3mn}}$ to a radical quantity of the second degree.

Ans. $\sqrt{ab^2c^3}$.

6. Reduce $\sqrt[n]{a^{\frac{n}{2}}}$ to a radical quantity of the second degree.

Ans. $\sqrt{a}$.

55.—1. Reduce $\sqrt{2}$ and $\sqrt[4]{3}$ to equivalent expressions having equal indices.

Ans. $\sqrt[4]{4}$ and $\sqrt[4]{3}$.

2. Reduce $\sqrt{a}$ and $\sqrt[3]{b}$ to equivalent expressions having equal indices.

Ans. $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$.

3. Reduce $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c}$, $\sqrt[5]{d}$, and $\sqrt[6]{e}$ to equivalent expressions having equal indices.

Ans. $\sqrt[60]{a^{30}}$, $\sqrt[60]{b^{20}}$, $\sqrt[60]{c^{15}}$, $\sqrt[60]{d^{12}}$, $\sqrt[60]{e^{10}}$.

4. Reduce $\sqrt{a^3}$, $\sqrt[3]{b^2}$, and $\sqrt[4]{c^3}$ to equivalent expressions having equal indices.

Ans. $\sqrt[12]{a^{18}}$, $\sqrt[12]{b^8}$, $\sqrt[12]{c^{15}}$.

5. Reduce $\sqrt[6]{a^2}$, $\sqrt[9]{b^3}$, $\sqrt[12]{c^4}$, $\sqrt[2n]{d^n}$, and $\sqrt[n]{e^{\frac{n}{2}}}$ to equivalent expressions having equal indices.

Ans. $\sqrt[2]{a}$, $\sqrt[2]{b}$, $\sqrt[2]{c}$, $\sqrt[2]{d}$, $\sqrt[2]{e}$.

6. Reduce $(a-b)^{\frac{m}{n}}$, $(c+d)^{\frac{n}{m}}$, and $(c-d)^{\frac{p}{q}}$ to equivalent expressions having equal indices.

Ans. $(a-b)^{\frac{m^2q}{mnq}}$, $(c+d)^{\frac{n^2q}{mnq}}$, $(c-d)^{\frac{mnp}{mnq}}$.

COMBINATIONS OF RADICAL QUANTITIES.

56.—1. Find the sum of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

Ans. $6a\sqrt{x}$.

2. Find the sum of $\sqrt{128}$ and $\sqrt{72}$. *Ans.* $14\sqrt{2}$.

3. Find the sum of $\sqrt[3]{135}$ and $\sqrt[3]{40}$. *Ans.* $5\sqrt[3]{5}$.

4. Find the sum of $9\sqrt[3]{4}$ and $\sqrt[3]{108}$. *Ans.* $12\sqrt[3]{4}$.

5. Find the sum of $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{2}{9}}$. *Ans.* $\frac{5}{6}\sqrt{2}$.

6. Find the sum of $3\sqrt{a^2b}$ and $3\sqrt{16a^4b}$.

Ans. $(12a^2 + 3a)\sqrt{b}$.

7. Find the sum of $a(3a^2b)^{\frac{1}{3}}$ and $(12a^4b)^{\frac{1}{3}}$.

Ans. $3a^2(3b)^{\frac{1}{3}}$.

8. Find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$.

Ans. $3\frac{1}{10}\sqrt{6}$.

9. Find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.

Ans. $6\frac{3}{4}\sqrt[3]{2}$.

10. Find the sum of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

Ans. $\left(\frac{a}{2} + \frac{2x^2}{3}\right)\sqrt{b}$.

11. Find the sum of $3\sqrt[3]{a^3bc}$ and $5a\sqrt[3]{b^3c^3}$.

Ans. $8a\sqrt[3]{bc}$.

12. Find the sum of $\sqrt[3]{54a^{n+6}b^3}$, $\sqrt[6]{256a^{2n-6}b^{12}}$, $\sqrt[3]{8a^{12n+27}}$, and $\sqrt[12]{16c^{12}a^{4n}}$.

Ans. $\left(3a^2b + \frac{2b^2}{a} + a^{n+3} + c\right)\sqrt[3]{2a^n}$.

13. Find the sum of $\sqrt{a^{-1}}$, $b\sqrt{\frac{1}{a}}$, and $c\sqrt{\frac{1}{a^{-3}}}$.

Ans. $\left(\frac{1+b}{a} + ac\right)\sqrt{a}$.

14. Find the sum of $a^{\frac{3}{5}}\sqrt{a^{\frac{3}{5}} + b^{\frac{3}{5}}}$ and $b\left(1+\frac{a^{\frac{3}{5}}}{b^{\frac{3}{5}}}\right)^{\frac{1}{2}}$.

Ans. $\sqrt{a^3 + 3a^{\frac{4}{3}}b^{\frac{2}{3}} + 3a^{\frac{2}{3}}b^{\frac{4}{3}} + b^2}$.

57.—1. Subtract $\sqrt{8}$ from $\sqrt{18}$. *Ans.* $\sqrt{2}$.

2. Subtract $\sqrt{48ax^2}$ from $\sqrt{108ax^2}$. *Ans.* $2x\sqrt{3a}$.

3. Subtract $6\sqrt[3]{2}$ from $6\sqrt[3]{4}$. *Ans.* $6\sqrt[3]{4} - 6\sqrt[3]{2}$.

4. Subtract $\sqrt[3]{24}$ from $\sqrt[3]{192}$. *Ans.* $2\sqrt[3]{3}$.

5. Subtract $\sqrt{\frac{1}{6}}$ from $\sqrt{\frac{8}{27}}$. *Ans.* $\frac{1}{18}\sqrt{6}$.

6. Subtract $2\sqrt{\frac{1}{10}}$ from $3\sqrt{\frac{2}{5}}$. *Ans.* $\frac{2}{5}\sqrt{10}$.

7. Subtract $\frac{2}{5}\sqrt{\frac{1}{6}}$ from $\frac{3}{4}\sqrt{\frac{2}{3}}$. *Ans.* $\frac{11}{60}\sqrt{6}$.

8. Simplify $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$.

Ans. $\sqrt{2}$.

9. Simplify $8\left(\frac{3}{4}\right)^{\frac{1}{2}} + \frac{1}{2} \times 12^{\frac{1}{2}} - \frac{4}{3} \times 27^{\frac{1}{2}} - 2\left(\frac{3}{16}\right)^{\frac{1}{2}}$.

Ans. $\frac{1}{2}\sqrt{3}$.

10. Simplify $4\sqrt{147} - 3\sqrt{75} - 6\sqrt{\frac{1}{3}}$. *Ans.* $11\sqrt{3}$.

11. Subtract $3\left(\frac{1}{3}\right)^{\frac{1}{3}}$ from $72^{\frac{1}{3}}$. *Ans.* $\sqrt[3]{9}$.

12. Simplify $b\,(8a^6b)^{\frac{1}{3}} + 4a\,(a^3b^4)^{\frac{1}{3}} - (125a^6b^4)^{\frac{1}{3}}$.

Ans. $a^2b^{\frac{4}{3}}$.

13. Subtract $\frac{2b}{3}\left(\frac{a^6}{27b^5}\right)^{\frac{1}{3}}$ from $\frac{3}{4}(8a^3b)^{\frac{1}{3}}$.

Ans. $\left(\frac{3}{2}a - \frac{2a^2}{9b}\right)b^{\frac{1}{3}}$.

14. Subtract $\sqrt[3]{54a^{n+6}b^3}$ from $\sqrt[6]{256a^{2n-6}b^{12}}$.

Ans. $\left(\frac{2b^2}{a} - 3a^2b\right)\sqrt[3]{2a^n}$.

58.—1. Multiply $5\sqrt{5}$ by $3\sqrt{8}$. *Ans.* $30\sqrt{10}$.

2. Multiply $4\sqrt{12}$ by $3\sqrt{2}$. *Ans.* $24\sqrt{6}$.

3. Multiply $3\sqrt{2}$ by $2\sqrt{8}$. *Ans.* 24.

4. Multiply $2\sqrt[3]{14}$ by $3\sqrt[3]{4}$. *Ans.* $12\sqrt[3]{7}$.

5. Multiply $\frac{1}{2}\sqrt{5}$ by $\frac{2}{3}\sqrt{2}$. *Ans.* $\frac{1}{3}\sqrt{10}$.

6. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$. *Ans.* $\sqrt[6]{a^3b^2}$.

7. Multiply $4\sqrt{3}$ by $3\sqrt[3]{4}$. *Ans.* $12\sqrt[6]{432}$.

8. Multiply $5\sqrt{a}$ by $3\sqrt[3]{a}$. *Ans.* $15\sqrt[6]{a^5}$.

9. Find the product of $\sqrt{2a^{-1}}$, $\sqrt[3]{2ab^{-1}}$, $\sqrt[4]{2abc^{-1}}$, and $\sqrt[5]{2^{-1}a^{-1}b^{-1}c^{-1}}$.

Ans. $\frac{1}{abc}\sqrt[60]{2^{53}a^{58}b^{43}c^{38}}$.

10. Multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$. *Ans.* $(a+b)^{\frac{13}{12}}$.

11. Multiply $(a-b)^{\frac{1}{m}}$ by $(a-b)^{-\frac{1}{m}}$. *Ans.* 1.

12. Find the product of $ax^{\frac{1}{n}}$, $by^{\frac{1}{m}}$, and $cz^{\frac{1}{p}}$.

Ans. $abc\sqrt[mnp]{x^{mp}y^{np}z^{mn}}$.

59.—1. Multiply $1-\sqrt[3]{a^4}\sqrt[4]{x^{-3}}+\sqrt[3]{a}\sqrt[4]{x}-\sqrt[3]{a^{-2}}\sqrt[4]{x^3}$ by $\sqrt[4]{a^3}\sqrt{x^{-1}}$.

Ans. $\sqrt[4]{a^2x^{-2}}-\sqrt[12]{a^{25}x^{-15}}+\sqrt[12]{a^{12}x^{-8}}-\sqrt[12]{ax^3}$.

2. Multiply $x^{\frac{4}{3}}-2(xy)^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

Ans. $x^2-3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}-y^2$.

3. Multiply $(xy^2)^{\frac{1}{3}}+(xy^3)^{\frac{1}{4}}+(xy^4)^{\frac{1}{5}}$ by $(x^2y)^{\frac{1}{3}}-(x^4y)^{\frac{1}{5}}$.

Ans. $x^{\frac{11}{12}}y^{\frac{13}{12}}+x^{\frac{13}{15}}y^{\frac{17}{15}}-x^{\frac{21}{20}}y^{\frac{19}{20}}-x^{\frac{17}{15}}y^{\frac{13}{15}}$.

4. Multiply $\frac{1}{2}\sqrt{a}+\frac{1}{3}\sqrt[3]{a}+\frac{1}{4}\sqrt[4]{a}$ by $\sqrt[4]{a}-\frac{3}{5}\sqrt[5]{a}$.

Ans. $\frac{1}{2}\sqrt[4]{a^3}-\frac{3}{10}\sqrt[10]{a^7}+\frac{1}{3}\sqrt[12]{a^7}-\frac{1}{5}\sqrt[15]{a^8}+\frac{1}{4}\sqrt{a}-\frac{3}{20}\sqrt[20]{a^9}$.

5. Multiply $(a^{-\frac{1}{2}})^{\frac{1}{3}} + \left\{(a^{\frac{1}{2}}b)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$ by $(a^{-\frac{1}{2}})^{\frac{1}{3}} - \left\{(a^{\frac{1}{2}}b)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$. *Ans.* $a^{-\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{3}}$.

6. Multiply $a^{\frac{5}{6}} + a^{\frac{1}{2}}x^{-\frac{1}{3}} + a^{\frac{3}{6}}x^{-\frac{1}{4}} + a^{\frac{1}{4}}x^{-\frac{2}{3}} + a^{\frac{1}{6}}x^{-\frac{1}{2}} + x^{-\frac{5}{6}}$ by $a^{\frac{3}{6}} - a^{\frac{1}{4}}x^{-\frac{1}{6}} + a^{\frac{1}{6}}x^{-\frac{1}{4}} - x^{-\frac{3}{6}}$.

Ans. $a + a^{\frac{3}{4}}x^{-\frac{1}{4}} - a^{\frac{1}{4}}x^{-\frac{3}{4}} - x^{-1}$.

7. Find the product of $(a+b)^{\frac{1}{m}}$, $(a+b)^{\frac{1}{n}}$, $(a-b)^{\frac{1}{m}}$, and $(a-b)^{\frac{1}{n}}$. *Ans.* $(a^2-b^2)^{\frac{m+n}{mn}}$.

8. Find the product of $4+2\sqrt{2}$, $1-\sqrt{3}$, $4-2\sqrt{2}$, $\sqrt{2}+\sqrt{3}$, $1+\sqrt{3}$, and $\sqrt{2}-\sqrt{3}$. *Ans.* 16.

9. Find the product of $x-1+2^{\frac{1}{2}}$, $x-1-2^{\frac{1}{2}}$, $x+2+3^{\frac{1}{2}}$, and $x+2-3^{\frac{1}{2}}$.

Ans. $x^4 + 2x^3 - 8x^2 - 6x - 1$.

60.—1. Divide $4\sqrt{50}$ by $2\sqrt{5}$. *Ans.* $2\sqrt{10}$.

2. Divide $6\sqrt[3]{100}$ by $3\sqrt[3]{5}$. *Ans.* $2\sqrt[3]{20}$.

3. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

4. Divide $\sqrt{7}$ by $\sqrt[3]{7}$. *Ans.* $\sqrt[6]{7}$.

5. Divide $2\sqrt[5]{100}$ by $3\sqrt[5]{80}$. *Ans.* $\frac{1}{3}\sqrt[5]{40}$.

6. Divide $\frac{5}{14}\sqrt[3]{\frac{2}{3}}$ by $\frac{5}{21}\sqrt[3]{\frac{9}{4}}$. *Ans.* 1.

7. Divide $\frac{1}{10}\sqrt[3]{\frac{5}{9}}$ by $\frac{9}{5}\sqrt{\frac{5}{6}}$.

Ans. $\frac{1}{270}\sqrt[6]{2^3 \times 5^5 \times 3^5}$.

8. Divide $15\sqrt[3]{x^4}$ by $35\sqrt[4]{x}$. *Ans.* $\frac{3}{7}x^{\frac{13}{12}}$.

9. Divide $(x^3y^{mn})^{\frac{1}{m}}$ by $(x^2y^{mn})^{\frac{1}{n}}$. *Ans.* $x^{\frac{3n-2m}{mn}}y^{n-m}$.

10. Divide $(a+b)^{\frac{1}{n}}$ by $(a-b)^{-\frac{1}{n}}$. *Ans.* $(a^2-b^2)^{\frac{1}{n}}$.

61.—1. Divide $\sqrt{x^5} - 4\sqrt{x^3} - 2\sqrt{x} + 6x - x^2$ by $\sqrt{x^3} + 2 - 4\sqrt{x}$. *Ans.* $x - \sqrt{x}$.

2. Divide $\left(\frac{4}{3}a^{\frac{3}{2}}\right)^{\frac{1}{2}} - \left(\frac{3}{4}a^{\frac{3}{2}}\right)^{\frac{1}{2}}$ by $3^{\frac{3}{2}}a^{-\frac{1}{2}}$. *Ans.* $\frac{1}{18}a^{\frac{5}{4}}$.

3. Divide $16x - y^3$ by $2x^{\frac{1}{4}} - y^{\frac{1}{2}}$.

Ans. $8x^{\frac{3}{4}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 2x^{\frac{1}{4}}y + y^{\frac{3}{2}}$.

4. Divide $16x - \frac{1}{16}y^4$ by $2x^{\frac{1}{4}} - \frac{1}{2}y$.

Ans. $8x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y + \frac{1}{2}x^{\frac{1}{4}}y^2 + \frac{1}{8}y^3$.

5. Divide $a^2 - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

Ans. $a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{3}} + ab^{\frac{1}{3}} + a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{5}{3}}$.

6. Divide $a^3 - b^3$ by $a^{\frac{3}{4}} + b^{\frac{3}{4}}$.

Ans. $a^{\frac{9}{4}} - a^{\frac{3}{2}}b^{\frac{3}{4}} + a^{\frac{3}{4}}b^{\frac{3}{2}} - b^{\frac{9}{4}}$.

7. Divide $x^2 - 16y^2$ by $x^{\frac{1}{2}} - 2y^{\frac{1}{2}}$.

Ans. $x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 4x^{\frac{1}{2}}y + 8y^{\frac{3}{2}}$.

8. Divide $a^{\frac{5}{3}} + a^2b^{\frac{1}{3}} + a^{\frac{4}{3}}b^{\frac{2}{3}} + ab + a^{\frac{1}{3}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

Ans. $a^2 + ab^{\frac{2}{3}} + b^{\frac{4}{3}}$.

9. Divide $x^3 - 2(x^3y^3)^{\frac{1}{2}} - x^2(x^3y^3)^{\frac{1}{6}} + 2y^{\frac{13}{6}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{3}}$.

Ans. $x^{\frac{5}{2}} - 2x^{\frac{1}{2}}y^{\frac{3}{2}} - 2y^{\frac{11}{6}}$.

10. Divide $a - b^2$ by $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}$.

Ans. $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

11. Divide $x^{-1} - y^{-1}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.

Ans. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

12. Divide $x^{-3} - 64y^2$ by $x^{-\frac{1}{2}} + 2y^{\frac{1}{3}}$.

Ans. $x^{-\frac{1}{2}} - 2x^{-2}y^{\frac{1}{3}} + 4x^{-\frac{3}{2}}y^{\frac{2}{3}} - 8x^{-1}y + 16x^{-\frac{1}{2}}y^{\frac{4}{3}} - 32y^{\frac{5}{3}}$.

INVOLUTION OF RADICAL QUANTITIES.

62.—1. Find the square of $5\sqrt{a}$. *Ans.* $25a$.

2. Find the square of $3\sqrt[3]{a^2}$. *Ans.* $9a\sqrt[3]{a}$.

3. Find the 3d power of $-2\sqrt[12]{x^6y^4z^8}$.

Ans. $-8xy\sqrt[4]{x^2z^8}$.

4. Find the 4th power of $\frac{2}{3}\sqrt[3]{\frac{3x^2}{4y}}$. *Ans.* $\frac{2x^2}{27y^2}\sqrt[3]{6x^2y^2}$.

5. Find the 7th power of $\left\{a^3b\,(a^3bc)^{\frac{1}{3}}\right\}^{\frac{1}{3}}$.

Ans. $a^{\frac{21}{3}}b^{\frac{7}{3}}c^{\frac{7}{30}}$.

6. Find the 4th power of $a^{\frac{1}{3}}x^{-1}$. *Ans.* $a^{\frac{4}{3}}x^{-4}$.

7. Find the 6th power of $2x\,(3y)^{\frac{1}{3}}$. *Ans.* $1728x^6y^8$.

8. Find the 3d power of $\frac{2}{3}x^{\frac{6}{6}}y^{-\frac{1}{6}}$. *Ans.* $\frac{8}{27}x^{\frac{6}{3}}y^{-\frac{1}{3}}$.

9. Find the m^{th} power of $(a+b)^{\frac{1}{n}}(a+b)^{-\frac{1}{n}}$.

Ans. 1.

10. Find the m^{th} power of $\left(a^{\frac{1}{n}}+b^{\frac{1}{n}}\right)^{\frac{1}{n}}\left(a^{\frac{1}{n}}-b^{\frac{1}{n}}\right)^{\frac{1}{n}}$.

Ans. $\left(a^{\frac{2}{n}}-b^{\frac{2}{n}}\right)^{\frac{m}{n}}$.

63.—1. Find the square of $\sqrt{a}+b\sqrt{c}$.

Ans. $a+2ab\sqrt{c}+b^2c$.

2. Find the 4th power of $\sqrt{a}-\sqrt{x}$.

Ans. $a^2-4\sqrt{a^3x}+6ax-4\sqrt{ax^3}+x^2$.

3. Find the 5th power of $a^{\frac{1}{3}}+x^{\frac{1}{3}}$.

Ans. $a^{\frac{5}{3}}+5a^{\frac{4}{3}}x^{\frac{1}{3}}+10ax^{\frac{2}{3}}+10a^{\frac{2}{3}}x+5a^{\frac{1}{3}}x^{\frac{4}{3}}+x^{\frac{5}{3}}$.

4. Find the 5th power of $\frac{x^{\frac{1}{3}}}{2}+\frac{y^{\frac{1}{3}}}{3}$.

Ans. $\frac{x^{\frac{5}{3}}}{32}+\frac{5x^2y^{\frac{1}{3}}}{48}+\frac{5x^{\frac{3}{3}}y^{\frac{2}{3}}}{36}+\frac{5xy}{54}+\frac{5x^{\frac{1}{3}}y^{\frac{4}{3}}}{162}+\frac{y^{\frac{5}{3}}}{243}$.

5. Find the 3d power of $a^{\frac{1}{3}}x^{-1} + a^{-\frac{1}{3}}x$.

Ans. $ax^{-3} + 3a^{\frac{1}{3}}x^{-1} + 3a^{-\frac{1}{3}}x + a^{-1}x^3$.

6. Find the 3d power of $2x\,(3y)^{\frac{1}{3}} - 5x^{\frac{1}{3}}y$.

Ans. $24x^3y\,(3y)^{\frac{1}{3}} - 180x^{\frac{5}{3}}y^2 + 150x^2y^2\,(3y)^{\frac{1}{3}} - 125x^{\frac{3}{3}}y^3$.

7. Find the 3d power of $a^{\frac{1}{3}} + b^{\frac{1}{3}} - c^{\frac{1}{3}}$.

Ans. $a + b - c + 3\,(a^2b)^{\frac{1}{3}} - 3\,(a^2c)^{\frac{1}{3}} + 3\,(ab^2)^{\frac{1}{3}} + 3(ac^2)^{\frac{1}{3}} - 3\,(b^2c)^{\frac{1}{3}} + 3\,(bc^2)^{\frac{1}{3}} - 6\,(abc)^{\frac{1}{3}}$.

8. Find the 4th power of $2x^{\frac{3}{2}} - 3y^{-\frac{3}{4}}$.

Ans. $16x^6 - 96x^{\frac{9}{2}}y^{-\frac{3}{4}} + 216x^3y^{-\frac{3}{2}} - 216x^{\frac{3}{2}}y^{-\frac{9}{4}} + 81y^{-3}$.

EVOLUTION OF RADICAL QUANTITIES.

64.—1. Find the square root of $\sqrt[6]{a^4}$. *Ans.* $\sqrt[3]{a}$.

2. Find the square root of $3\sqrt[3]{5}$. *Ans.* $\sqrt[6]{135}$.

3. Find the cube root of $a^3\sqrt[5]{b^3}$. *Ans.* $a\sqrt[5]{b}$.

4. Find the cube root of $(27a^3x)^{\frac{1}{3}}$. *Ans.* $(3ax^{\frac{1}{3}})^{\frac{1}{3}}$.

5. Find the cube root of $(27a^3x)^{\frac{1}{3}}$. *Ans.* $(3ax^{\frac{1}{3}})^{\frac{1}{3}}$.

6. Find the m^{th} root of $2^{\frac{1}{m}}a^mb^{2m}c^2$. *Ans.* $2^{\frac{1}{m^2}}ab^2c^{\frac{2}{m}}$.

7. Find the 15th root of $\sqrt[5]{a^{30}b^{27}c^{18}}$. *Ans.* $^{3\,5}\sqrt{a^{10}b^9c^6}$.

8. Find the mn^{th} root of $\sqrt[n]{a^{2m}b^{3m}c^{4m}}$. *Ans.* $\sqrt[n^2]{a^2b^3c^4}$.

65.—1. Find the square root of $4\sqrt{x} + 8\sqrt[4]{xy} + 4\sqrt{y}$.

Ans. $2\sqrt[4]{x} + 2\sqrt[4]{y}$.

2. Find the square root of $-2 + a^{2\sqrt{2}} + a^{-2\sqrt{2}}$.

Ans. $a^{\sqrt{2}} - a^{-\sqrt{2}}$.

3. Find the square root of $5x^2 - 4x(5cx)^{\frac{1}{2}} + 4c$.

Ans. $5^{\frac{1}{2}}x^{\frac{3}{2}} - 2c^{\frac{1}{2}}$.

4. Find the square root of $1 + \frac{41}{16}a - \frac{3+3a}{2}a^{\frac{1}{2}} + a^2$.

Ans. $1 - \frac{3}{4}a^{\frac{1}{2}} + a$.

5. Find the cube root of $\frac{1}{8}a^3 - \frac{3}{2}a^2\sqrt{b} + 6ab - 8\sqrt{b^3}$.

Ans. $\frac{1}{2}a - 2\sqrt{b}$.

6. Find the cube root of $a^{-\frac{3}{2}}x^{\frac{3}{2}} - 3a^{-1}x + 6a^{-\frac{1}{2}}x^{\frac{1}{2}} - 7 + 6a^{\frac{1}{2}}x^{-\frac{1}{2}} - 3ax^{-1} + a^{\frac{3}{2}}x^{-\frac{3}{2}}$. *Ans.* $a^{-\frac{1}{2}}x^{\frac{1}{2}} - 1 + a^{\frac{1}{2}}x^{-\frac{1}{2}}$.

7. Find the cube root of $54a - 32x - 36(2ax)^{\frac{1}{3}}(3a^{\frac{1}{3}} - 2^{\frac{4}{3}}x^{\frac{1}{3}})$.

Ans. $3(2a)^{\frac{1}{3}} - 2(4x)^{\frac{1}{3}}$.

8. Find the cube root of $\{x + (a^2x)^{\frac{1}{3}}\}^2 + \{a + (ax^2)^{\frac{1}{3}}\}^2$.

Ans. $x^{\frac{2}{3}} + a^{\frac{2}{3}}$.

9. Find the fourth root of $16a^6 - 96a^{\frac{9}{2}}x^{\frac{3}{4}} + 216a^3x^{\frac{3}{2}} - 216a^{\frac{3}{2}}x^{\frac{9}{4}} + 81x^3$. *Ans.* $2a^{\frac{3}{2}} - 3x^{\frac{3}{4}}$.

REDUCTION OF FRACTIONS HAVING SURD DENOMINATORS TO EQUIVALENT ONES HAVING RATIONAL DENOMINATORS.

66. Reduce each of the following fractions to an equivalent one having a rational denominator:

1. $\frac{2}{\sqrt{3}}$. *Ans.* $\frac{2\sqrt{3}}{3}$.

2. $\frac{3}{5\sqrt[3]{4}}$. *Ans.* $\frac{3\sqrt[3]{2}}{10}$.

3. $\frac{\sqrt[4]{2}}{2\sqrt[4]{3}}$. *Ans.* $\frac{\sqrt[4]{54}}{6}$.

4. $\frac{a}{b\sqrt{c}}$. *Ans.* $\frac{a\sqrt{c}}{bc}$.

5. $\frac{a}{b\sqrt[3]{c}}$. *Ans.* $\frac{a\sqrt[3]{c^2}}{bc}$.

6. $\frac{a}{b\sqrt[4]{c}}$. *Ans.* $\frac{a\sqrt[4]{c^3}}{bc}$.

7. $\frac{a}{b\sqrt[4]{c^2}}$. *Ans.* $\frac{a\sqrt[4]{c^2}}{bc}$.

8. $\frac{\sqrt{a}}{b\sqrt[5]{c^4}}$. *Ans.* $\frac{\sqrt[10]{a^5c^2}}{bc}$.

9. $\frac{\sqrt[3]{a}}{b\sqrt[3]{a^2}}$. *Ans.* $\frac{\sqrt[3]{a^2}}{ab}$.

10. $\frac{a}{b\sqrt[m]{c}}$. *Ans.* $\frac{a\sqrt[m]{c^{m-1}}}{bc}$.

11. $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$. *Ans.* $\frac{a^{\frac{1}{3}}b^{\frac{2}{3}}}{b}$.

12. $\frac{a^{\frac{1}{m}}}{b^{\frac{1}{n}}}$. *Ans.* $\frac{a^{\frac{1}{m}}b^{\frac{n-1}{n}}}{b}$.

67. Reduce each of the following fractions to an equivalent one having a rational denominator:

1. $\frac{1+\sqrt{3}}{2\sqrt{2}-3\sqrt{3}}$. *Ans.* $-\frac{9+2\sqrt{2}+3\sqrt{3}+2\sqrt{6}}{19}$.

2. $\frac{1}{2\sqrt{2}-\sqrt{3}}$. *Ans.* $\frac{2\sqrt{2}+\sqrt{3}}{5}$.

3. $\dfrac{3}{\sqrt{5}-\sqrt{2}}$. *Ans.* $\sqrt{5}+\sqrt{2}$.

4. $\dfrac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$. *Ans.* $9+2\sqrt{15}$.

5. $\dfrac{31}{4\sqrt{5}+3\sqrt{2}}$. *Ans.* $\dfrac{4\sqrt{5}-3\sqrt{2}}{2}$.

6. $\dfrac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$. *Ans.* $\dfrac{4+3\sqrt{2}-2\sqrt{3}-\sqrt{6}}{4}$.

7. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. *Ans.* $\dfrac{3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{12}$.

8. $\dfrac{1}{\sqrt{3}+\sqrt[3]{5}}$.

Ans. $\dfrac{9\sqrt{3}-9\sqrt[3]{5}+3\sqrt[6]{16875}-15+5\sqrt[6]{675}-5\sqrt[3]{25}}{2}$.

9. $\dfrac{2-\sqrt[3]{3}}{2+\sqrt[3]{3}}$. *Ans.* $\dfrac{5-8\sqrt[3]{3}+4\sqrt[3]{9}}{11}$.

10. $\dfrac{\sqrt[3]{9}-\sqrt{2}}{\sqrt[3]{9}+\sqrt{2}}$

Ans. $17-12\sqrt{2}+12\sqrt[3]{3}+8\sqrt[3]{9}-8\sqrt[6]{72}-6\sqrt[6]{648}$.

11. $\dfrac{1}{a^3-b^{\frac{1}{3}}}$. *Ans.* $\dfrac{a^6+a^3b^{\frac{1}{3}}+b^{\frac{2}{3}}}{a^9-b}$.

12. $\dfrac{x}{3+\sqrt{x}}$. *Ans.* $\dfrac{3x-x\sqrt{x}}{9-x}$.

13. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$. *Ans.* $\dfrac{a^2+b-2a\sqrt{b}}{a^2-b}$.

14. $\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}$. *Ans.* $\dfrac{a+(a^2-x^2)^{\frac{1}{2}}}{x}$.

15. $\dfrac{x}{x+\sqrt{1+x^2}}$. *Ans.* $x\sqrt{1+x^2}-x^2$.

16. $\dfrac{x}{\sqrt{a^2+x^2}-x}$. *Ans.* $\dfrac{x\sqrt{a^2+x^2}+x^2}{a^2}$.

17. $\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$. *Ans.* $2x^2+1-2x\sqrt{x^2+1}$.

18. $\dfrac{a\sqrt{b}+c\sqrt{d}}{a\sqrt{b}-c\sqrt{d}}$. *Ans.* $\dfrac{a^2b+2ac\sqrt{bd}+c^2d}{a^2b-c^2d}$.

SIMPLIFICATION OF COMPLEX RADICAL QUANTITIES.

68. Simplify each of the following expressions:

1. $\sqrt{a+\sqrt[3]{b^3}}$. *Ans.* $\sqrt{a+b}$.

2. $\sqrt{7+2\sqrt{10}}$. *Ans.* $\pm\sqrt{5}\pm\sqrt{2}$.

3. $\sqrt{7+4\sqrt{3}}$. *Ans.* $\pm 2\pm\sqrt{3}$.

4. $\sqrt{2+\sqrt{3}}$. *Ans.* $\pm\dfrac{1}{\sqrt{2}}(\sqrt{3}+1)$.

5. $\sqrt{16 + 6\sqrt{7}}$. *Ans.* $\pm 3 \pm \sqrt{7}$.

6. $\sqrt{28 - 10\sqrt{3}}$. *Ans.* $\pm 5 \mp \sqrt{3}$.

7. $\sqrt{8 - 2\sqrt{15}}$. *Ans.* $\pm \sqrt{5} \mp \sqrt{3}$.

8. $\sqrt{41 - 12\sqrt{5}}$. *Ans.* $\pm 6 \mp \sqrt{5}$.

9. $\sqrt{37 - 20\sqrt{3}}$. *Ans.* $\pm 5 \mp 2\sqrt{3}$.

10. $\sqrt{4 - \sqrt{7}}$. *Ans.* $\pm \frac{1}{\sqrt{2}}(\sqrt{7} - 1)$.

11. $\sqrt{4\frac{1}{3} - \frac{4}{3}\sqrt{3}}$. *Ans.* $\pm 2 \mp \frac{1}{3}\sqrt{3}$.

12. $\sqrt{3\sqrt{6} + 2\sqrt{12}}$. *Ans.* $\sqrt[4]{6}(\pm 1 \pm \sqrt{2})$.

13. $\sqrt{3\sqrt{3} + 2\sqrt{6}}$. *Ans.* $\sqrt[4]{3}(\pm 1 \pm \sqrt{2})$.

14. $\sqrt{\sqrt{18} - \sqrt{16}}$. *Ans.* $\sqrt[4]{2}(\pm \sqrt{2} \mp 1)$.

15. $\sqrt{8\sqrt{3} - 6\sqrt{5}}$. *Ans.* $\sqrt[4]{3}(\pm \sqrt{5} \mp \sqrt{3})$.

16. $\sqrt{\frac{1}{2}\sqrt{18} + 2}$. *Ans.* $\frac{1}{\sqrt[4]{2}}(\pm \sqrt{2} \pm 1)$.

17. $\sqrt{\frac{8}{7}\sqrt{21}-2\sqrt{3}}$. *Ans.* $\sqrt[4]{21}\left(\pm 1 \mp \frac{1}{7}\sqrt{7}\right)$.

18. $\sqrt{a^2+2x\sqrt{a^2-x^2}}$. *Ans.* $\pm x \pm \sqrt{a^2-x^2}$.

19. $\sqrt{2a+2\sqrt{a^2-b^2}}$. *Ans.* $\pm\sqrt{a+b} \pm \sqrt{a-b}$.

20. $\sqrt{4x^3+xy^2-y^3-4xy\sqrt{x^2-xy}}$.

Ans. $\pm 2x\sqrt{x} \mp y\sqrt{x-y}$.

21. $\sqrt{1+\sqrt{1-m^2}}$. *Ans.* $\pm\sqrt{\frac{1+m}{2}} \pm \sqrt{\frac{1-m}{2}}$.

22. $\sqrt{2+2(1-x)\sqrt{1+2x-x^2}}$.

Ans. $\pm(1-x) \pm \sqrt{1+2x-x^2}$.

23. $\sqrt{x+y+z+2\sqrt{xz+yz}}$.

Ans. $\pm\sqrt{x+y} \pm \sqrt{z}$.

24. $\left\{x^m+\frac{1}{4}(b^p x^{2m})^{\frac{2}{np}} - b^{\frac{1}{n}} x^{\frac{mp+4}{2p}}\right\}^{\frac{1}{2}}$. *Ans.* $x^{\frac{m}{2}} - \frac{1}{2}b^{\frac{1}{n}}x^{\frac{2}{p}}$.

25. $\left\{a^{-\frac{2}{xy}} + \left(a^{\frac{1}{y}}b\right)^{\frac{1}{r}} - 2b^{\frac{1}{2r}}a^{\frac{x-2r}{2rxy}}\right\}^{\frac{1}{2}}$. *Ans.* $a^{-\frac{1}{xy}} - a^{\frac{1}{2ry}}b^{\frac{1}{2r}}$.

26. $\sqrt[3]{\frac{1}{8}a^3 - \frac{3}{2}a^2\sqrt{b} + 6ab - 8\sqrt{b^3}}$.

Ans. $\frac{1}{2}a - 2\sqrt{b}$.

27. $\sqrt{3\sqrt[3]{5}}$. *Ans.* $\sqrt[6]{135}$.

28. $\sqrt[3]{\sqrt{27a^3x}}$. *Ans.* $\sqrt[6]{27a^3x}$.

29. $\sqrt[3]{\sqrt[3]{27a^3x}}$. *Ans.* $\sqrt[9]{27a^3x}$.

30. $\sqrt[m]{\sqrt[n]{\sqrt[p]{ab}}}$. *Ans.* $\sqrt[mnp]{ab}$.

IMAGINARY QUANTITIES.

COMBINATIONS OF SIMPLE IMAGINARY QUANTITIES.

69.—1. Find the sum of $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$. *Ans.* $12\sqrt{-1}$.

2. Find the sum of $\sqrt[4]{-1}$, $\sqrt[4]{-16}$, and $\sqrt[4]{-81}$.
Ans. $6\sqrt[4]{-1}$.

3. Find the sum of $\sqrt{-2}$, $\sqrt{-3}$, and $\sqrt{-4}$.
Ans. $(\sqrt{2}+\sqrt{3}+2)\sqrt{-1}$.

4. Find the sum of $\sqrt{-a}$, $\sqrt{-b}$, $\sqrt{-c}$, and $\sqrt{-d}$. *Ans.* $(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})\sqrt{-1}$.

5. Find the sum of $(-1)^{\frac{1}{2}}$, $\left(-\frac{1}{4}\right)^{\frac{1}{2}}$, and $\left(-\frac{1}{9}\right)^{\frac{1}{2}}$.
Ans. $\frac{11}{6}(-1)^{\frac{1}{2}}$.

70.—1. Subtract $\sqrt{-4}$ from $9\sqrt{-1}$.

Ans. $7\sqrt{-1}$.

2. Subtract $\sqrt[4]{-16}$ from $2\sqrt[4]{-81}$.

Ans. $4\sqrt[4]{-1}$.

3. Subtract $3\sqrt{-8}$ from $9\sqrt{-10}$.

Ans. $(9\sqrt{10} - 6\sqrt{2})\sqrt{-1}$.

4. Subtract $a\sqrt{-b}$ from $c\sqrt{-d}$.

Ans. $(c\sqrt{d} - a\sqrt{b})\sqrt{-1}$.

5. Subtract $\dfrac{a^{\frac{1}{2}}}{(-1)^{-\frac{1}{2}}}$ from $\dfrac{(-1)^{\frac{1}{2}}}{b^{-\frac{1}{2}}}$.

Ans. $(b^{\frac{1}{2}} - a^{\frac{1}{2}})(-1)^{\frac{1}{2}}$.

71.—1. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

Ans. $-6\sqrt{6}$.

2. Multiply $8\sqrt{-9}$ by $-10\sqrt{-16}$. *Ans.* 960.

3. Multiply $a\sqrt{-bc}$ by $m\sqrt{-n}$.

Ans. $-am\sqrt{bcn}$.

4. Multiply $a\sqrt{-(1+x)^2}$ by $-b\sqrt{-(1-x)^2}$.

Ans. $ab(1-x^2)$.

5. Multiply $(-a)^{-\frac{1}{2}}$ by $(-b)^{-\frac{1}{2}}$. *Ans.* $-\dfrac{1}{\sqrt{ab}}$.

72.—1. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.

2. Divide $2\sqrt{-7}$ by $-3\sqrt{-5}$. *Ans.* $-\frac{2}{15}\sqrt{35}$.

3. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$. *Ans.* $\frac{\sqrt{3}}{18}$.

4. Divide $a\sqrt{-(x+1)^2}$ by $b\sqrt{-(x+1)^4}$.

Ans. $\frac{a}{b(x+1)}$.

5. Divide $(a\sqrt[4]{-b})^2$ by $(c\sqrt[4]{-d})^2$.

Ans. $\frac{a^2\sqrt{bd}}{c^2d}$.

MISCELLANEOUS EXAMPLES IN IMAGINARY QUANTITIES.

73.—1. Multiply $5+2\sqrt{-3}$ by $2-\sqrt{-3}$.

Ans. $16-\sqrt{-3}$.

2. Multiply $x-a\sqrt{-1}$ by $x+a\sqrt{-1}$.

Ans. x^2+a^2.

3. Multiply $x+\frac{a}{2}(\sqrt{3}-\sqrt{-1})$ by $x-\frac{a}{2}(\sqrt{3}-\sqrt{-1})$.

Ans. $x^2-\frac{a^2}{2}+\frac{a^2}{2}\sqrt{-3}$.

4. Reduce $(x-a\sqrt{-1})\left\{x+\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}\times\left\{x+\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}(x+a\sqrt{-1})\left\{x-\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}\left\{x-\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}$ to its simplest form.

Ans. x^6+a^6.

5. Simplify $\frac{1}{x-1}+\frac{2}{2x+1-\sqrt{-3}}+\frac{2}{2x+1+\sqrt{-3}}$.

Ans. $\frac{3x^2}{x^3-1}$.

6. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

Ans. $1+\sqrt{-2}$.

7. Divide $3+2\sqrt{-1}$ by $3-2\sqrt{-1}$.

Ans. $\frac{5+12\sqrt{-1}}{13}$.

8. Divide 40 by $6+\sqrt{-4}$. *Ans.* $6-\sqrt{-4}$.

9. Find the third power of $\sqrt[3]{x+\sqrt{-y^2}}-\sqrt[3]{x-\sqrt{-y^2}}$.

Ans. $2y\sqrt{-1}-3\sqrt[3]{x^2+y^2}\left(\sqrt[3]{x+y\sqrt{-1}}-\sqrt[3]{x-y\sqrt{-1}}\right)$.

10. Simplify $\sqrt{2n\sqrt{-1}}$. *Ans.* $(\pm 1\pm\sqrt{-1})\sqrt{n}$.

11. Simplify $\sqrt{1-4\sqrt{-3}}$. *Ans.* $\pm 2\mp\sqrt{-3}$.

12. Simplify $\sqrt{2\sqrt{-3}-2}$. *Ans.* $\pm 1\mp\sqrt{-3}$.

13. Simplify $\sqrt{4\sqrt{-5}-1}$. *Ans.* $\pm 2\mp\sqrt{-5}$.

14. Simplify $\sqrt{-3+2\sqrt{2}}$.

Ans. $\pm\sqrt{-1}\mp\sqrt{-2}$.

15. Simplify $\sqrt{21 - \sqrt{-400}}$. *Ans.* $\pm 5 \mp 2\sqrt{-1}$.

16. Simplify $\sqrt{-3 - 4\sqrt{-1}}$. *Ans.* $\mp 1 \pm 2\sqrt{-1}$.

17. Simplify $\sqrt{.03 + .04\sqrt{-1}}$. *Ans.* $\pm .2 \pm .1\sqrt{-1}$.

18. Simplify $\sqrt{4mn + 2(m^2 - n^2)\sqrt{-1}}$.

Ans. $\pm (m + n) \pm (m - n)\sqrt{-1}$.

19. Simplify $\sqrt{\sqrt{-79 - 8\sqrt{-5}}}$.

Ans. $\pm 2 \mp \sqrt{-5}$.

20. Express $\sqrt[4]{-1}$ in the form of $a \pm b\sqrt{-1}$.

Ans. $\pm \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}\sqrt{-1}$.

21. Express $\sqrt[3]{-1}$ in the form of $a \pm b\sqrt{-1}$.

Ans. $\frac{1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{-1}$.

22. Express $\frac{a + b\sqrt{-1}}{c + d\sqrt{-1}}$ in the form of $A + B\sqrt{-1}$.

Ans. $\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\sqrt{-1}$.

23. If $2 + y + x\sqrt{-8} = 8 + x + y\sqrt{-2}$, show that $x = 6$ and $y = 12$.

RADICAL EQUATIONS.

74. Find the value of x in each of the following equations:

1. $\sqrt{12+x} = 2 + \sqrt{x}$. *Ans.* $x = 4$.

2. $\sqrt{5x+10} = \sqrt{5x} + 2$. *Ans.* $x = \frac{9}{20}$.

3. $(\sqrt{9x}+5)^2 = 108 + 9x$. *Ans.* $x = \left(\frac{83}{30}\right)^2$.

4. $\frac{\sqrt[3]{17x-6}}{3} + \frac{3}{8} = \frac{25}{24}$. *Ans.* $x = \frac{14}{17}$.

5. $\frac{3x-1}{\sqrt{3x}+1} = 1 + \frac{\sqrt{3x}-1}{2}$. *Ans.* $x = 3$.

6. $\frac{ax-b^2}{\sqrt{ax}+b} - \frac{\sqrt{ax}-b}{c} = c$.

Ans. $x = \frac{1}{a}\left(b + \frac{c^2}{c-1}\right)^2$.

7. $\sqrt{x} - \sqrt{a+x} = \sqrt{\frac{a}{x}}$. *Ans.* $x = \frac{\sqrt{a}}{\sqrt{a}+2}$.

8. $\frac{1}{a}\sqrt{a+x} + \frac{1}{x}\sqrt{a+x} = \frac{1}{b}\sqrt{x}$.

Ans. $x = \frac{a\sqrt[3]{b^2}}{\sqrt[3]{a^2} - \sqrt[3]{b^2}}$.

9. $a + x \pm \sqrt{a^2+x^2} = b$. *Ans.* $x = \frac{b}{2}\left(\frac{b-2a}{b-a}\right)$.

10. $x^{\frac{1}{2}} + \left\{x - (1-x)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = 1.$ *Ans.* $x = \frac{16}{25}$.

11. $x^{\frac{1}{2}} + \left\{a - (ax + x^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = a^{\frac{1}{2}}.$ *Ans.* $x = \frac{9a}{16}$.

12. $(a-x)^{\frac{1}{2}} + 2(a+x)^{\frac{1}{2}} = \left\{a - x + (ax + x^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$

Ans. $x = \frac{64a}{1025}$.

13. $x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} = \frac{na}{(a+x)^{\frac{1}{2}}}.$ *Ans.* $x = \frac{(n-1)^2}{2n-1}a$.

14. $(1-x)^{\frac{1}{2}} + \left\{1 - x + (1+x)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = (1+x)^{\frac{1}{2}}.$

Ans. $x = \frac{24}{25}$.

15. $\dfrac{a^{\frac{1}{2}} - \left\{a - (a^2 - ax)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}{a^{\frac{1}{2}} + \left\{a - (a^2 - ax)^{\frac{1}{2}}\right\}^{\frac{1}{2}}} = b.$

Ans. $x = a\left\{1 - \left(\frac{2b^{\frac{1}{2}}}{1+b}\right)^4\right\}$.

16. $\dfrac{1 + x + (2x + x^2)^{\frac{1}{2}}}{1 + x - (2x + x^2)^{\frac{1}{2}}} = a\dfrac{(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}}}{(2+x)^{\frac{1}{2}} - x^{\frac{1}{2}}}.$

Ans. $x = \frac{1}{2}(a^{\frac{1}{2}} - a^{-\frac{1}{2}})^2$.

17. $\sqrt{x} = \sqrt{2} + \sqrt{x-2}.$ *Ans.* $x = 2$.

18. $\sqrt{x-5} - 7 + \sqrt{x-12} = 0.$ *Ans.* $x = 21$.

19. $\dfrac{1}{\sqrt{x}} - \sqrt{x} = \sqrt{x+1}.$ *Ans.* $x = \frac{1}{3}$.

20. $\sqrt{x} - \sqrt{\frac{a}{x}} = \sqrt{a+x}$. *Ans.* $x = \frac{\sqrt{a}}{\sqrt{a}+2}$.

21. $\left(\frac{a^2}{x}+b\right)^{\frac{1}{2}} - \left(\frac{a^2}{x}-b\right)^{\frac{1}{2}} = c$. *Ans.* $x = \frac{4a^2c^2}{4b^2+c^4}$.

22. $\frac{x-ax}{\sqrt{x}} = \frac{\sqrt{x}}{x}$. *Ans.* $x = \frac{1}{1-a}$.

23. $(a+x)^{\frac{1}{m}} = (x^2+5ax+b^2)^{\frac{1}{2m}}$. *Ans.* $x = \frac{a^2-b^2}{3a}$.

24. $\frac{\sqrt{a+\sqrt{x}}}{\sqrt[3]{x}} + \frac{\sqrt{a-\sqrt{x}}}{\sqrt[3]{x}} = \sqrt[6]{x}$. *Ans.* $x=4(a-1)$.

QUADRATIC EQUATIONS WITH ONE UNKNOWN QUANTITY.—QUADRATIC EXPRESSIONS.

INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

75. Solve the following equations:

1. $9x^2 + 9 = 3x^2 + 63$. *Ans.* $x = \pm 3$.

2. $3x(x-8) = x(24-5x)$. *Ans.* $x = 0$ or 6.

3. $3x^2 + 8 = 56$. *Ans.* $x = \pm 4$.

4. $(x+2)^2 = 4x + 5$. *Ans.* $x = \pm 1$.

5. $2x^2 - \frac{1}{2}x^2 = x(5x-3)$. *Ans.* $x = 0$ or $\frac{6}{7}$.

6. $8\left(x+\frac{1}{x}\right) = \frac{21+70x}{7} - 3$. *Ans.* $x = \pm 2$.

7. $\frac{3}{1+x} + \frac{3}{1-x} = 8$. *Ans.* $x = \pm\frac{1}{2}$.

8. $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$. *Ans.* $x=\pm 8$.

9. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$. *Ans.* $x=0$ or 4.

10. $\frac{x}{12}+\frac{x^2-15}{5x}=\frac{x}{5}$. *Ans.* $x=\pm 6$.

11. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=2\cdot\frac{x+3}{x-3}$. *Ans.* $x=0$ or $\frac{4}{3}$.

12. $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x$.

Ans. $x=\pm\sqrt{3}$.

13. $\frac{\sqrt{1+x}}{1+\sqrt{1+x}}=\frac{\sqrt{1-x}}{1-\sqrt{1-x}}$. *Ans.* $x=\pm\frac{\sqrt{3}}{2}$.

14. $\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x+2a}+\sqrt{x-2a}}=\frac{x}{2a}$. *Ans.* $x=\pm 2a$.

15. $x^2+\frac{2}{3}x^2-\frac{1}{2}x^2=x$. *Ans.* $x=0$ or $\frac{6}{7}$.

16. $3ax^2-10ax=8ax+ax^2$. *Ans.* $x=0$ or 9.

17. $\frac{x+2}{2x+1}=\frac{x+3}{3x+1}$. *Ans.* $x=\pm 1$.

18. $\frac{x-1}{x}-\frac{3x}{x-1}=2$. *Ans.* $x=\pm\frac{1}{2}$.

19. $4x+\frac{16}{x}=0$. *Ans.* $x=\pm 2\sqrt{-1}$.

20. $\dfrac{x+7}{x(x-7)} - \dfrac{x-7}{x(x+7)} = \dfrac{7}{x^2-73}$. *Ans.* $x = \pm 9$.

21. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}} = 4$. *Ans.* $x = \pm \dfrac{5}{8}$.

22. $\dfrac{a}{x} + \dfrac{\sqrt{a^2-x^2}}{x} = \dfrac{x}{b}$. *Ans.* $x = \pm \sqrt{2ab-b^2}$.

23. $\dfrac{1}{\sqrt{1-x}+1} + \dfrac{1}{\sqrt{1+x}-1} = \dfrac{1}{x}$. *Ans.* $x = \pm \dfrac{\sqrt{3}}{2}$.

24. $\dfrac{1+x^3}{(1+x)^2} + \dfrac{1-x^3}{(1-x)^2} = a$. *Ans.* $x = \pm \sqrt{\dfrac{a-2}{a+4}}$.

PROBLEMS.

76.—1. Find two numbers, one of which is three times the other, and the difference of whose squares is 128.

Ans. ± 4 and ± 12.

2. Find two numbers, one of which is $\frac{4}{5}$ of the other, and the difference of whose squares is 81.

Ans. ± 12 and ± 15.

3. Find two such numbers that their difference is $\frac{3}{8}$ of the greater, and the sum of their squares is 356.

Ans. ± 10 and ± 16.

4. The difference between two numbers is 10, and the square of the less is $\frac{9}{25}$ of the square of the greater. What are the numbers? *Ans.* 15 and 25.

5. A tailor bought a certain number of yards of broadcloth for \$324, paying $\frac{4}{3}$ as many dollars per yard as there were yards. How many yards did he buy? *Ans.* 27.

6. A and B start at the same time from P and Q respectively and travel toward each other. When they meet, it is found that A has traveled 30 miles more than B, and that A will reach Q in 4 days, and B will reach P in 9 days, after they meet. Find the distance between P and Q.

Ans. 150 miles.

7. A sets off from London to York, and B at the same time from York to London, and they travel uniformly. A reaches York 16 hours, and B reaches London 36 hours, after they met on the road. In what time does each perform the journey? *Ans.* A in 40 hours, B in 60 hours.

8. An army was marching with 5 men more in depth than in front; but on the appearance of the enemy the front was increased by 845 men, and the whole army was thus drawn up in 5 lines; find the number of men. *Ans.* 4550.

9. Two persons, A and B, set out from different places to meet each other. They started at the same time, and traveled on the direct road between the two places. On meeting, it appeared that A had traveled a miles more than B; and that A could have traveled B's distance in m days, but that B would have been n days in traveling A's distance. Find the distance between the two places.

Ans. $a\dfrac{\sqrt{m}+\sqrt{n}}{\sqrt{n}-\sqrt{m}}$ miles.

10. Divide the number d into two such parts that the square of the quotient obtained by dividing the first by the second shall be equal to $\dfrac{a}{b}$.

Ans. $d\left(\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{b}}\right)$ and $d\left(\dfrac{\sqrt{b}}{\sqrt{a}\pm\sqrt{b}}\right)$.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

77. Solve the following equations:

1. $x^2 + 6x = 40$. *Ans.* $x = 4$ or -10.
2. $x^2 + 8x = 65$. *Ans.* $x = 5$ or -13.
3. $x^2 + 4x = 5$. *Ans.* $x = 1$ or -5.
4. $x^2 + 10x = 56$. *Ans.* $x = 4$ or -14.
5. $x^2 - 8x - 48 = 0$. *Ans.* $x = 12$ or -4.
6. $x^2 + 12x - 108 = 0$. *Ans.* $x = 6$ or -18.
7. $x^2 - 6x = 16$. *Ans.* $x = 8$ or -2.
8. $x^2 - 4x - 3 = 9$. *Ans.* $x = 6$ or -2.
9. $x^2 + 6x + 80 = 600$. *Ans.* $x = 20$ or -26.
10. $x^2 + 6x - 135 = 0$. *Ans.* $x = 9$ or -15.
11. $x^2 - x = 2$. *Ans.* $x = 2$ or -1.
12. $x^2 + x = 306$. *Ans.* $x = 17$ or -18.
13. $x^2 - 16x + 15 = 0$. *Ans.* $x = 15$ or 1.
14. $x^2 + 12x + 50 = 0$. *Ans.* $x = -6 \pm \sqrt{-14}$.
15. $x^2 - 5x = 24$. *Ans.* $x = 8$ or -3.
16. $x^2 - 11x = 12$, *Ans.* $x = 12$ or -1.
17. $x^2 + 17x = 84$. *Ans.* $x = 4$ or -21.
18. $x^2 + 21x = 100$. *Ans.* $x = 4$ or -25.
19. $x^2 - 7x + 1 = 9$. *Ans.* $x = 8$ or -1.
20. $x^2 + 3x - 28 = 0$. *Ans.* $x = 4$ or -7.
21. $x^2 - \frac{3}{4}x = \frac{85}{4}$. *Ans.* $x = 5$ or $-4\frac{1}{4}$.
22. $x^2 + \frac{5}{3}x = 14$. *Ans.* $x = 3$ or $-4\frac{2}{3}$.

23. $2x^2 - 24 = 22x$. *Ans.* $x = 12$ or -1.

24. $3x^2 - 30 = 27x$. *Ans.* $x = 10$ or -1.

25. $x^2 - 2x + 2 = 0$. *Ans.* $x = 1 \pm \sqrt{-1}$.

26. $x^2 - 8ax = 3b^2$. *Ans.* $x = 4a \pm \sqrt{3b^2 + 16a^2}$.

27. $3x^2 - 5x = 28$. *Ans.* $x = 4$ or $-2\frac{1}{3}$.

28. $x^2 - 20x - a^2 = 0$. *Ans.* $x = 10 \pm \sqrt{100 + a^2}$.

29. $2x^2 + 5x - 7 = 0$. *Ans.* $x = 1$ or $-3\frac{1}{2}$.

30. $4x^2 - 20x + 25 = 0$. *Ans.* $x = \frac{5}{2}$.

31. $x^2 - \frac{2}{3}x + 14\frac{1}{4} = 16$. *Ans.* $x = 1\frac{1}{2}$ or $-\frac{5}{6}$.

32. $2x^2 - 26x = 28$. *Ans.* $x = 14$ or -1.

33. $5x^2 + 4x = 25$. *Ans.* $x = \frac{-2 \pm \sqrt{129}}{5}$.

34. $6x^2 - 13x + 6 = 0$. *Ans.* $x = \frac{3}{2}$ or $\frac{2}{3}$.

35. $4x^2 - 7x = 492$. *Ans.* $x = 12$ or $-10\frac{1}{4}$.

36. $x^2 + 5ax - c^2 = 0$. *Ans.* $x = \frac{-5a \pm \sqrt{25a^2 + 4c^2}}{2}$.

37. $x(2x + 1) - (x + 8)(x + 3) = 0$.

Ans. $x = 12$ or -2.

38. $(2x - 3)^2 = 8x$. *Ans.* $x = 4\frac{1}{2}$ or $\frac{1}{2}$.

39. $(x - 1)(x - 2) = 0$. *Ans.* $x = 2$ or 1.

40. $4(x^2 - 1) = 4x - 1$. *Ans.* $x = \frac{3}{2}$ or $-\frac{1}{2}$.

41. $11x^2 + 1 = 4(2 - x)^2$. *Ans.* $x = \frac{5}{7}$ or -3.

42. $(x - 12)(x + 2) = 0$. *Ans.* $x = 12$ or -2.

43. $a + bx + cx^2 = 0$. *Ans.* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$.

44. $x^2 - 16(x - 4) = 1$. *Ans.* $x = 9$ or 7.

45. $x(3 - x) - 2 = 0$. *Ans.* $x = 2$ or 1.

46. $7(x^2 + 6x) = 42x - (3x + 2)$.
Ans. $x = \frac{-3 \pm \sqrt{-47}}{14}$.

47. $10x^2 - 9ax + 9a^2 = 0$. *Ans.* $x = \frac{9a \pm a\sqrt{-279}}{20}$.

48. $acx^2 - bcx + adx - bd = 0$. *Ans.* $x = \frac{b}{a}$ or $-\frac{d}{c}$.

49. $(x - 3)^2 + 4x = 44$. *Ans.* $x = 7$ or -5.

50. $(x + 3a)(x + 4a) = 72a^2$. *Ans.* $x = 5a$ or $-12a$.

51. $b(a - x)^2 = cx^2$. *Ans.* $x = \frac{-ab \pm a\sqrt{bc}}{c - b}$.

52. $x^3 + (19 - x)^3 = 1843$. *Ans.* $x = 11$ or 8.

53. $(a - b)x = ab - x^2$. *Ans.* $x = b$ or $-a$.

54. $(x + a)^2 = 5ax - (x - a)^2$. *Ans.* $x = 2a$ or $\frac{a}{2}$.

55. $(a + x)(a - x) = b^2 - 2x$.
Ans. $x = 1 \pm \sqrt{a^2 - b^2 + 1}$.

56. $13x - 29 - (x + 2)\{x - 1 - 6(x - 9)\} = 0$.
Ans. $x = 9$ or -3.

57. $(x-5)(x-3) - (x-5)(2x-5) - (x+7)(x-2) = 0.$

Ans. $x = 2$ or -1.

58. $cx + \frac{ac}{a+b} = (a+b)x^2$. *Ans.* $x = \frac{c \pm \sqrt{4ac+c^2}}{2(a+b)}$.

59. $48(x+4) + 48(x-3) = 27(x-3)(2x+8)$.

Ans. $x = 4$ or $-\frac{29}{9}$.

60. $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{3}\right) + \left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right) = \left(x-\frac{1}{4}\right)\left(x-\frac{1}{5}\right)$.

Ans. $x = \frac{2}{3}$ or $\frac{3}{10}$.

61. $\frac{3}{4}(x^2-3) = \frac{1}{8}(x-3)$. *Ans.* $x = \frac{5}{3}$ or $-\frac{3}{2}$.

62. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}$. *Ans.* $x = 2$ or -3.

63. $\frac{10}{x} - \frac{14-2x}{x^2} = \frac{22}{9}$. *Ans.* $x = 3$ or $\frac{21}{11}$.

64. $\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\frac{1}{3}$. *Ans.* $x = 6$ or $3\frac{1}{13}$.

65. $\frac{x^3-10x^2+1}{x^2-6x+9} - (x-3) = 0$. *Ans.* $x = 1$ or -28.

66. $8x + 11 + \frac{7}{x} = \frac{21+65x}{7}$. *Ans.* $x = 7$ or $-\frac{7}{9}$.

67. $\frac{x}{5} + \frac{5}{x} = 5\frac{1}{5}$. *Ans.* $x = 25$ or 1.

68. $\frac{8-x}{2} - \frac{2x-11}{x-3} = \frac{x-2}{6}$. *Ans.* $x = 6$ or $\frac{1}{2}$.

69. $\frac{x+3}{2}+\frac{16-2x}{2x-5}=5\frac{1}{2}$. *Ans.* $x=5$ or $6\frac{9}{10}$.

70. $\frac{x^2-4}{7}+\frac{9x+3}{16}=\frac{9x-3}{21}+\frac{(x-1)^2}{4}$.

Ans. $x=5$ or $\frac{11}{12}$.

71. $\frac{x+11}{x}=7-\frac{9+4x}{x^2}$. *Ans.* $x=3$ or $-\frac{1}{2}$.

72. $\frac{x}{x+60}=\frac{7}{3x-5}$. *Ans.* $x=14$ or -10.

73. $\frac{x}{x-6}-\frac{70}{5-4x}=0$. *Ans.* $x=\frac{-65\pm\sqrt{10945}}{8}$.

74. $\frac{4x}{x+3}-\frac{x-3}{2x+5}=2$. *Ans.* $x=3$ or $-\frac{7}{3}$.

75. $\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}$. *Ans.* $x=36$ or 12.

76. $\frac{3x-5}{9x}-\frac{6x}{3x-25}=\frac{1}{3}$. *Ans.* $x=\frac{25}{18}$ or $-\frac{5}{3}$.

77. $\frac{2x-4}{4}=3-\frac{9-x}{x-3}$. *Ans.* $x=7$ or 6.

78. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$. *Ans.* $x=1\pm\sqrt{1-a^2}$.

79. $(a+b)x^2+(a-b)x=\frac{ab}{a+b}$.

Ans. $x=\frac{b}{a+b}$ or $-\frac{a}{a+b}$.

80. $\dfrac{1}{x+a}+\dfrac{1}{x+2a}+\dfrac{1}{x+3a}=\dfrac{3}{x}$.

Ans. $x=\dfrac{-11\pm\sqrt{13}}{6}a$.

81. $\dfrac{2}{x^{-1}}+\dfrac{1}{(4x+8)^{-\frac{1}{2}}}=\dfrac{7}{2}$. *Ans.* $x=4\frac{1}{4}$ or $\frac{1}{4}$.

82. $\sqrt{(4+x)(5-x)}=2x-10$. *Ans.* $x=5$ or $3\frac{1}{5}$.

83. $\sqrt{x+a}-\sqrt{x+b}=\sqrt{2x}$.

Ans. $x=-\dfrac{a+b}{2}\pm\sqrt{\dfrac{a^2+b^2}{2}}$.

84. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}$.

Ans. $x=9a$ or $-a$.

85. $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$. *Ans.* $x=8$ or $-\frac{8}{9}$.

86. $x\pm\sqrt{10x+6}=9$. *Ans.* $x=25$ or 3.

87. $\sqrt{a+x}\cdot\sqrt{b-x}=c$.

Ans. $x=\dfrac{b-a\pm\sqrt{(a+b)^2-4c^2}}{2}$.

88. $\dfrac{x}{\sqrt{x}+\sqrt{a-x}}+\dfrac{x}{\sqrt{x}-\sqrt{a-x}}=\dfrac{b}{\sqrt{x}}$.

Ans. $x=\dfrac{b\pm\sqrt{b^2-2ab}}{2}$.

89. $\dfrac{a+x}{\sqrt{a}+\sqrt{a+x}}-\dfrac{a-x}{\sqrt{a}-\sqrt{a+x}}=0$.

Ans. $x=\dfrac{a}{2}(1\pm\sqrt{5})$.

78. PROBLEMS.

1. A company dining at a hotel had to pay $3.50; but before the bill was presented two of the company went away; in consequence of which, those who remained had to pay each 20 cents more than if all had been present. How many persons were in the company? *Ans.* 7.

2. Find a number such that, if it be subtracted from 22, and the remainder multiplied by the number, the product shall be 117. *Ans.* 13 or 9.

3. A man traveled 36 miles at a uniform rate. If his rate had been one mile more per hour, he would have traveled the same distance in three hours less time. At what rate did he travel? *Ans.* 3 miles per hour.

4. Divide the number 60 into two such parts that their product shall be 704. *Ans.* 44 and 16.

5. A man starts from the foot of a mountain to walk to its top. His rate of walking during the second half of the distance is half a mile per hour less than that during the first half, and he reaches the top in $5\frac{1}{2}$ hours. He descends by the same route in $3\frac{3}{4}$ hours, walking one mile per hour faster than during the first half of the ascent. Find the rate of descent. *Ans.* 4 miles per hour.

6. The distance from P to Q is 247 miles. A started from P toward Q, and, at the same time, B started from Q toward P. A's rate of travel was 9 miles per day, and when he met B it was found that the number of days they had been traveling was greater by 3 than the number of miles in B's rate. How far had each traveled at the time of meeting?

Ans. A, 117 miles; B, 130 miles.

7. A tailor bought a piece of cloth for \$147, from which he cut 12 yards for his own use; he then sold the remainder for \$120.25 and gained \$0.25 per yard. How many yards were there in the piece? *Ans.* 49.

8. The distance from P to Q is d miles. A started from P toward Q, and, at the same time, B started from Q toward P. A's rate of travel was a miles per day, and when he met B it was found that the number of days they had been traveling was greater by b than the number of miles in B's rate. How far had each traveled at the time of meeting?

Ans. $\begin{cases} \text{A, } \dfrac{a\left[b-a+\sqrt{4d+(b-a)^2}\right]}{2} \text{ miles;} \\ \text{B, } d-\dfrac{a\left[b-a+\sqrt{4d+(b-a)^2}\right]}{2} \text{ miles.} \end{cases}$

9. The distance from P to Q is 320 miles. A started from P toward Q, and, at the same time, B started from Q toward P. A's rate of travel per day was 8 miles more than that of B, and when they met it was found that the number of days they had been traveling was equal to half the number of miles in B's rate. How many days did they travel before they met? *Ans.* 8.

10. Two couriers, A and B, started at the same time to a place 90 miles distant. A, whose rate of travel was one mile per hour greater than that of B, reached the end of his journey one hour before him. At what rate did each travel?

Ans. A 10 miles, and B 9 miles, per hour.

11. Find two numbers whose sum is a, and the sum of whose squares is b.

Ans. $\dfrac{a\pm\sqrt{2b-a^2}}{2}$ and $\dfrac{a\mp\sqrt{2b-a^2}}{2}$.

12. Find two numbers whose sum is 41, and the sum of whose squares is 901. *Ans.* 15 and 26.

13. Find two numbers whose difference is 8, and the sum of whose squares is 544.

Ans. 12 and 20, or -12 and -20.

14. Find two numbers whose difference is a, and the sum of whose squares is b.

Ans. $\frac{a \pm \sqrt{2b - a^2}}{2}$ and $\frac{-a \pm \sqrt{2b - a^2}}{2}$.

15. What are eggs worth per dozen when a more for b cents lowers the price c cents per dozen?

Ans. $\frac{1}{2}c + \sqrt{\frac{c^2}{4} + \frac{12bc}{a}}$ cents.

16. A started from C toward D, and traveled at the rate of 7 miles per hour. When he had traveled 4½ hours, B started from D toward C, and went every hour one-nineteenth of the distance from D to C. When B had traveled as many hours as he went miles in one hour, he met A. Find the distance from C to D.

Ans. 152 miles or 76 miles.

17. A man traveled a miles. If his rate of travel had been b miles per hour less, he would have been c hours longer in traveling the same distance. What was his rate of travel?

Ans. $\frac{b + \sqrt{\frac{4ab + b^2c}{c}}}{2}$ miles per hour.

18. The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and perpendicular is 3. Find the sides.

Ans. 15, 12, and 9.

19. If a plank 30 feet 4 inches long, 18 inches wide, and 3 inches thick, be made into a cubical box, what will be the length of one edge of the cube? *Ans.* 3 feet.

20. A and B bought a farm containing a acres, for which they paid \$$2b$, each paying \$$b$. A paid \$$d$ more per acre than B, in order to be permitted to take his share from the best land. How many acres did each get?

$$Ans. \begin{cases} \text{A, } \dfrac{2b + ad - \sqrt{4b^2 + a^2d^2}}{2d} \text{ acres;} \\ \text{B, } \dfrac{ad - 2b + \sqrt{4b^2 + a^2d^2}}{2d} \text{ acres.} \end{cases}$$

79.—1. Find the equation whose roots are 5 and 6.

Ans. $x^2 - 11x = -30$.

2. Find the equation whose roots are 4 and -5.

Ans. $x^2 + x = 20$.

3. Find the equation whose roots are -4 and 5.

Ans. $x^2 - x = 20$.

4. Find the equation whose roots are -2 and -3.

Ans. $x^2 + 5x = -6$.

5. Find the equation whose roots are $2\sqrt{3}$ and $3\sqrt{3}$.

Ans. $x^2 - 5x\sqrt{3} = -18$.

6. Find the equation whose roots are a and b.

Ans. $x^2 - (a + b)x = -ab$.

7. Find the equation whose roots are $2 + 3\sqrt{5}$ and $2 - 3\sqrt{5}$. *Ans.* $x^2 - 4x = 41$.

8. Find the equation whose roots are $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$. *Ans.* $x^2 - 4x = -5$.

9. Find the equation whose roots are $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$. *Ans.* $x^2 - 2ax = -(a^2 + b^2)$.

10. Resolve the first member of the equation $x^2 - 6x - 7 = 0$ into two binomial factors.

Ans. $(x - 7)(x + 1) = 0$.

11. Resolve the first member of the equation $x^2 - x - \frac{3}{4} = 0$ into two binomial factors.

Ans. $\left(x - \frac{3}{2}\right)\left(x + \frac{1}{2}\right) = 0$.

12. Resolve the first member of the equation $6x^2 - 13x + 6 = 0$ into three factors. *Ans.* $6\left(x - \frac{2}{3}\right)\left(x - \frac{3}{2}\right) = 0$.

13. Resolve the first member of the equation $\frac{5}{x^{-2}} - \frac{3}{x^{-1}} - 2x^0 = 0$ into three factors.

Ans. $5(x - 1)\left(x + \frac{2}{5}\right) = 0$.

14. Resolve the first member of the equation $x^2 - 2ax + a^2 + b^2 = 0$ into two factors.

Ans. $(x - a - b\sqrt{-1})(x - a + b\sqrt{-1}) = 0$.

QUADRATIC EXPRESSIONS.

80. Resolve each of the following expressions into its prime factors:

1. $x^2 - 10x - 24$. *Ans.* $(x - 12)(x + 2)$.
2. $x^2 + 2x - 80$. *Ans.* $(x - 8)(x + 10)$.
3. $x^2 - 18x + 32$. *Ans.* $(x - 16)(x - 2)$.

4. $x^2 - 13(x+6) + 10$. *Ans.* $(x-17)(x+4)$.

5. $x^2 - x - 11342$. *Ans.* $(x-107)(x+106)$.

6. $x^2 + 9x - 52$. *Ans.* $(x-4)(x+13)$.

7. $x^2 - 111x - 3400$. *Ans.* $(x-136)(x+25)$.

8. $x^2 - 5(x+89) - 5555$. *Ans.* $(x-80)(x+75)$.

9. $x^2 - \frac{9}{5}x + \frac{9}{20}$. *Ans.* $\left(x - \frac{3}{10}\right)\left(x - \frac{3}{2}\right)$.

10. $x^2 - \frac{5}{3}x - 4$. *Ans.* $(x-3)\left(x + \frac{4}{3}\right)$.

11. $2x^2 - 3x - 54$. *Ans.* $2(x-6)(x+4\frac{1}{2})$.

12. $4x^2 - 4x - 80$. *Ans.* $2 \cdot 2(x-5)(x+4)$.

13. $6x^2 + 5x - 4$. *Ans.* $3 \cdot 2\left(x - \frac{1}{2}\right)\left(x + \frac{4}{3}\right)$.

14. $12x^2 - x - 1740$. *Ans.* $3 \cdot 2 \cdot 2(x - 12\frac{1}{12})(x+12)$.

15. $3x^2 - 18x + 24$. *Ans.* $3(x-2)(x-4)$.

16. $11x^2 - 11\frac{1}{2} - 9x$. *Ans.* $11\left(x - \frac{3}{2}\right)\left(x + \frac{15}{22}\right)$.

17. $x^2 - (m-n)x - mn$. *Ans.* $(x-m)(x+n)$.

18. $\frac{1}{x^{-2}} + \frac{m+n}{x^{-1}} + mnx^0$. *Ans.* $(x+m)(x+n)$.

19. $x^2 - 9$. *Ans.* $(x+3)(x-3)$.

20. $x^2 + 9$. *Ans.* $(x + 3\sqrt{-1})(x - 3\sqrt{-1})$.

HIGHER EQUATIONS

WITH ONE UNKNOWN QUANTITY.

81. Solve the following equations:

1. $3x^{\frac{2}{3}} - x^{-\frac{2}{3}} + 2 = 0.$ *Ans.* $x = \left(\frac{1}{3}\right)^{\frac{3}{2}}$ or $(-1)^{\frac{3}{2}}$.

2. $x^6 - 7x^3 = 8.$ *Ans.* $x = 2$ or -1.

3. $11 - 9x^3 + x^6 = 299 + 3x^3 - 5x^6.$
Ans. $x = 2$ or $(-6)^{\frac{1}{3}}$.

4. $x^{-\frac{1}{3}} + 2 = \dfrac{x^{-1} + 8}{x^{-\frac{2}{3}} + 5}.$ *Ans.* $x = -8$ or $-\frac{1}{8}$.

5. $x^{\frac{7}{3}} + \dfrac{41x^{\frac{1}{3}}}{x} = \dfrac{97}{x^{\frac{2}{3}}} + x^{\frac{4}{3}}.$ *Ans.* $x = 4$ or $(-7)^{\frac{3}{2}}$.

6. $x^3 = 21 + (x^3 - 9)^{\frac{1}{2}}.$ *Ans.* $x = \pm 5$ or $\pm 3\sqrt{2}$.

7. $x^2 - 2x + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 11.$
Ans. $x = 1$ or $1 \pm 2\sqrt{15}$.

8. $x^2 - x + 5(2x^2 - 5x + 6)^{\frac{1}{2}} = \frac{1}{2}(3x + 33).$
Ans. $x = 3, -\frac{1}{2}$, or $\dfrac{5 \pm \sqrt{1329}}{4}$.

9. $9x - 3x^2 + 4(x^2 - 3x + 5)^{\frac{1}{2}} = 11.$
Ans. $x = \dfrac{3 \pm \sqrt{5}}{2}$ or $\dfrac{9 \pm \sqrt{-83}}{6}$.

10. $x + (x^2 - ax + b^2)^{\frac{1}{2}} = \frac{x^2}{a} + b.$

Ans. $x = a,\ 0,$ or $\frac{a}{2}\left\{1 \pm \left(5 - \frac{8b}{a}\right)^{\frac{1}{2}}\right\}.$

11. $\frac{(x^2 + x + 6)^{\frac{1}{2}}}{3} = \frac{20 - \frac{4}{3}(x^2 + x + 6)^{\frac{1}{2}}}{(x^2 + x + 6)^{\frac{1}{2}}}.$

Ans. $x = 5,\ -6,$ or $\frac{1}{2}(\pm\sqrt{377} - 1).$

12. $x^4 - 6x^2 = 27.$ *Ans.* $x = \pm 3$ or $\pm\sqrt{-3}.$

13. $\sqrt{x+1} - 2\sqrt[4]{x+1} = 4.$ *Ans.* $x = (1 \pm \sqrt{5})^4 - 1.$

14. $x^4 - 2x^3 + x = 132.$

Ans. $x = 4,\ -3,$ or $\frac{1 \pm \sqrt{-43}}{2}.$

15. $x^2 + \sqrt{x^2 + 11} = 31.$ *Ans.* $x = \pm\sqrt{38}$ or $\pm 5.$

16. $x^8 + 4x^4 = 12.$

Ans. $x = \pm\sqrt{\pm\sqrt{2}}$ or $\pm\sqrt{\pm\sqrt{-6}}.$

17. $x^{2n} - mx^n - p = 0.$ *Ans.* $x = \left(\frac{m \pm \sqrt{m^2 + 4p}}{2}\right)^{\frac{1}{n}}.$

18. $x^n - 2ax^{\frac{n}{2}} - b^2 = 0.$ *Ans.* $x = \left(a \pm \sqrt{a^2 + b^2}\right)^{\frac{2}{n}}.$

19. $\frac{4}{x^{-\frac{1}{3}}} + \frac{1}{x^{-\frac{1}{6}}} = 39x^0.$ *Ans.* $x = 729$ or $\left(-\frac{13}{4}\right)^6.$

82. *PROBLEMS.*

1. Some boys took 2916 apples from an orchard; and the number of apples that each took was equal to the square of twice the number of boys. How many boys were there? *Ans.* 9.

2. Find two numbers such, that the product of the greater and the square root of the less may be equal to 48, and the product of the less and square root of the greater may be 36. *Ans.* 16 and 9.

3. Find two numbers such, that the product of the less and the square of the greater may be equal to 448, and the product of the greater and the square of the less may be 392. *Ans.* 8 and 7.

4. Find a number such, that its cube may exceed 4 as much as 68 exceeds the cube of the square root of that number. *Ans.* 4.

5. Find a number such, that three times the square root of the remainder obtained by subtracting the fourth power of the number from 25 may be equal to $13\frac{1}{2}$ times the square of the number diminished by 18. *Ans.* 2.

6. Find a number such, that if seven times the cube root of its square be added to the cube root of its fourth power, the sum shall be equal to 44. *Ans.* $\pm$ 8.

SIMULTANEOUS EQUATIONS.

PAIRS OF EQUATIONS ONE OF WHICH IS OF THE FIRST AND THE OTHER OF THE SECOND DEGREE.

83. Solve the following pairs of equations:

1. $\left\{\begin{matrix} x+2y=7 \\ x^2+3xy-y^2=23 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=3 \text{ or } 15\frac{4}{11}, \\ y=2 \text{ or } -4\frac{1}{11}. \end{cases}$

2. $\left\{\begin{matrix} 2x+y=27 \\ 3xy=210 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=3\frac{1}{2} \text{ or } 10, \\ y=20 \text{ or } 7. \end{cases}$

3. $\left\{\begin{matrix} \dfrac{10x+y}{xy}=3 \\ y-x=2 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=2 \text{ or } -\dfrac{1}{3}. \\ y=4 \text{ or } \dfrac{5}{3}. \end{cases}$

4. $\left\{\begin{matrix} 2x-3y-1=0 \\ 2x^2+xy-5y^2=20 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=5 \text{ or } -9\frac{1}{4}, \\ y=3 \text{ or } -6\frac{1}{4}. \end{cases}$

5. $\left\{\begin{matrix} x+y=\dfrac{5}{4} \\ 8xy=3 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=\dfrac{1}{2} \text{ or } \dfrac{3}{4}. \\ y=\dfrac{3}{4} \text{ or } \dfrac{1}{2}. \end{cases}$

6. $\left\{\begin{matrix} ax+by=c \\ exy=f \end{matrix}\right\}$. *Ans.* $\begin{cases} x=\dfrac{c}{2a} \pm \dfrac{1}{2ae}\sqrt{c^2e^2-4abef}, \\ y=\dfrac{c}{2b} \mp \dfrac{1}{2be}\sqrt{c^2e^2-4abef}. \end{cases}$

7. $\left\{\begin{matrix} 2x+3y=14 \\ 5x^2-7y^2=52 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=4 \text{ or } -\dfrac{460}{17}, \\ y=2 \text{ or } \dfrac{386}{17}. \end{cases}$

8. $\left\{\begin{array}{l} x-y=2 \\ xy^{-1}-yx^{-1}=\frac{16}{15} \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5 \text{ or } \frac{3}{4}, \\ y=3 \text{ or } -\frac{5}{4}. \end{array}\right.$

9. $\left\{\begin{array}{l} 2x+3y=5 \\ 5x^2-2y^2=3 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=1 \text{ or } -2\frac{3}{37}. \\ y=1 \text{ or } 3\frac{2}{37}. \end{array}\right.$

10. $\left\{\begin{array}{l} xy^{-1}+yx^{-1}=8 \\ x+y=5 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{5\pm\sqrt{15}}{2}, \\ y=\frac{5\mp\sqrt{15}}{2}. \end{array}\right.$

11. $\left\{\begin{array}{l} x+y=4 \\ x^{-1}+y^{-1}=1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=2. \end{array}\right.$

12. $\left\{\begin{array}{l} 3x+2y=22 \\ 5x^2-3xy+y^2=45 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4 \text{ or } \frac{76}{47}, \\ y=5 \text{ or } \frac{403}{47}. \end{array}\right.$

13. $\left\{\begin{array}{l} x^2+y^2=202 \\ x+y=20 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=11 \text{ or } 9. \\ y=9 \text{ or } 11. \end{array}\right.$

14. $\left\{\begin{array}{l} x^2y^{-2}+4xy^{-1}=\frac{85}{9} \\ x-y=2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5 \text{ or } \frac{17}{10}, \\ y=3 \text{ or } -\frac{3}{10}. \end{array}\right.$

15. $\left\{\begin{array}{l} x^2+y^2=axy \\ x+y=b \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{b}{2}\left(1\pm\sqrt{\frac{a-2}{a+2}}\right), \\ y=\frac{b}{2}\left(1\mp\sqrt{\frac{a-2}{a+2}}\right). \end{array}\right.$

16. $\left\{\begin{matrix} x^2+y^2=34 \\ x-y=2 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5 \text{ or } -3, \\ y=3 \text{ or } -5. \end{matrix}\right.$

17. $\left\{\begin{matrix} x^2+y^2=a^2 \\ x+y=b \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\dfrac{b\pm\sqrt{2a^2-b^2}}{2}, \\ y=\dfrac{b\mp\sqrt{2a^2-b^2}}{2}. \end{matrix}\right.$

18. $\left\{\begin{matrix} x^2+3xy-y^2=23 \\ 3x+2y-13=0 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=3 \text{ or } 3\frac{18}{23}, \\ y=2 \text{ or } \frac{19}{23}. \end{matrix}\right.$

PARTICULAR SYSTEMS.

84. Solve the following pairs of equations:

1. $\left\{\begin{matrix} 4xy=96-x^2y^2 \\ x+y=6 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=4, 2, \text{ or } 3\pm\sqrt{21}. \\ y=2, 4, \text{ or } 3\mp\sqrt{21}. \end{matrix}\right.$

2. $\left\{\begin{matrix} 2x^2+3xy+y^2=20 \\ 5x^2+4y^2=41 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\pm 1 \text{ or } \pm 13\sqrt{\dfrac{41}{861}}, \\ y=\pm 3 \text{ or } \pm\sqrt{\dfrac{164}{861}}. \end{matrix}\right.$

3. $\left\{\begin{matrix} x^2-xy=6 \\ x^2+y^2=61 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 6 \text{ or } \pm\dfrac{\sqrt{2}}{2}, \\ y=\pm 5 \text{ or } \mp\dfrac{11\sqrt{2}}{2}. \end{matrix}\right.$

4. $\left\{\begin{matrix} x^2-xy=48 \\ xy-y^2=12 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 8, \\ y=\pm 2. \end{matrix}\right.$

5. $\left\{\begin{matrix} x^2+xy=a \\ y^2+xy=b \end{matrix}\right\}$. *Ans.* $\begin{cases} x=\pm\dfrac{a}{\sqrt{a+b}}, \\ y=\pm\dfrac{b}{\sqrt{a+b}}. \end{cases}$

6. $\left\{\begin{matrix} 6(x^2+y^2)=13xy \\ x^2-y^2=20 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=\pm 6, \\ y=\pm 4. \end{cases}$

7. $\left\{\begin{matrix} x^2+4xy+4y^2=256 \\ 3y^2-x^2=39 \end{matrix}\right\}$.

Ans. $\begin{cases} x=\pm 6 \text{ or } \pm 102. \\ y=\pm 5 \text{ or } \mp 59. \end{cases}$

8. $\left\{\begin{matrix} x^2+2xy+3y^2=17 \\ 2x^2+3xy+5y^2=28 \end{matrix}\right\}$.

Ans. $\begin{cases} x=\pm 1 \text{ or } \pm\frac{1}{2}\sqrt{2}, \\ y=\pm 2 \text{ or } \pm\frac{3}{2}\sqrt{2}. \end{cases}$

9. $\left\{\begin{matrix} x^{-1}+y^{-1}=x^0 \\ x^{-2}+y^{-2}=5 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=\frac{1}{2} \text{ or } -1, \\ y=-1 \text{ or } \frac{1}{2}. \end{cases}$

10. $\left\{\begin{matrix} x^2+xy-40=0 \\ 3xy-2y^2-27=0 \end{matrix}\right\}$.

Ans. $\begin{cases} x=\pm 5 \text{ or } \pm\frac{16}{10}\sqrt{10}. \\ y=\pm 3 \text{ or } \pm\frac{9}{10}\sqrt{10}. \end{cases}$

11. $\left\{\begin{matrix} xy^2+xy=24 \\ xy^3+x=56 \end{matrix}\right\}$. *Ans.* $\begin{cases} x=2 \text{ or } 54, \\ y=3 \text{ or } \frac{1}{3}. \end{cases}$

12. $\left\{\begin{matrix} x^2+2xy+y^2=81 \\ x^2-2xy+y^2=9 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 6, \\ y=\pm 3. \end{matrix}\right.$

13. $\left\{\begin{matrix} x^2+y^2-13=0 \\ 2xy-x-y=7 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=3,\ 2,\ \text{or}\ \pm\frac{1}{2}\sqrt{10}-2, \\ y=2,\ 3,\ \text{or}\ \mp\frac{1}{2}\sqrt{10}-2. \end{matrix}\right.$

14. $\left\{\begin{matrix} x^2-9y^2=28 \\ x^2-\ y^2=60 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 8, \\ y=\pm 2. \end{matrix}\right.$

15. $\left\{\begin{matrix} x^2+y^2=25 \\ xy=12 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 4, \\ y=\pm 3. \end{matrix}\right.$

16. $\left\{\begin{matrix} x^2-3xy+y^2+1=0 \\ 2x^2+2y^2-10=0 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=\pm 2 \text{ or } \pm 1, \\ y=\pm 1 \text{ or } \pm 2. \end{matrix}\right.$

17. $\left\{\begin{matrix} x^2(x-y)=25 \\ x^2(2x+3y)=550 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5, \\ y=4. \end{matrix}\right.$

18. $\left\{\begin{matrix} \dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}=\dfrac{3}{5} \\ y^2-x^2=5 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\pm 2, \\ y=\pm 3. \end{matrix}\right.$

19. $\left\{\begin{matrix} x^3-y^3=63 \\ x-y=3 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=4 \text{ or } -1, \\ y=1 \text{ or } -4. \end{matrix}\right.$

20. $\left\{\begin{matrix} x^3+y^3=152 \\ x^2y+xy^2=120 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5 \text{ or } 3, \\ y=3 \text{ or } 5. \end{matrix}\right.$

21. $\left\{\begin{matrix} x^3+y^3=351 \\ xy=14 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=7 \text{ or } 2, \\ y=2 \text{ or } 7. \end{matrix}\right.$

22. $\left\{\begin{matrix} x^4+y^4=257 \\ x+y=5 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=4,\ 1, \text{ or } \dfrac{5\pm\sqrt{-159}}{2}, \\ y=1,\ 4, \text{ or } \dfrac{5\mp\sqrt{-159}}{2}. \end{matrix}\right.$

23. $\left\{\begin{matrix} x^4+y^4=337 \\ xy=12 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\pm 4,\ \pm 3,\ \pm 4\sqrt{-1}, \text{ or } \pm 3\sqrt{-1}, \\ y=\pm 3,\ \pm 4,\ \pm 3\sqrt{-1}, \text{ or } \pm 4\sqrt{-1}. \end{matrix}\right.$

24. $\left\{\begin{matrix} x^{-1}+y^{-1}=a^{-1} \\ x^{-2}+y^{-2}=b^{-2} \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\dfrac{ab}{b^2-a^2}\left(b\pm\sqrt{2a^2-b^2}\right), \\ y=\dfrac{ab}{b^2-a^2}\left(b\mp\sqrt{2a^2-b^2}\right). \end{matrix}\right.$

25. $\left\{\begin{matrix} a^2x^2+b^2y^2=x^2y^2 \\ b^2y^4+a^2x^4=a^2b^2(x+y)^2 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\pm\sqrt{b\sqrt[3]{a^2b}+b^2}, \\ y=\pm\sqrt{a\sqrt[3]{ab^2}+a^2}. \end{matrix}\right.$

26. $\left\{\begin{matrix} a^2x^4+b^2y^4=a^2b^2(x+y)^2 \\ a^2x^2+b^2y^2=a^2b^2 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\pm\dfrac{b}{2\sqrt{a^2+b^2}}\left(\sqrt{a^2-4ab+b^2}\pm\sqrt{a^2+4ab+b^2}\right), \\ y=\pm\dfrac{a}{2\sqrt{a^2+b^2}}\left(\sqrt{a^2-4ab+b^2}\mp\sqrt{a^2+4ab+b^2}\right), \end{matrix}\right.$

27. $\left\{\begin{matrix} x^2+y^2=58 \\ x^3+y^3=370 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=7, \\ y=3. \end{matrix}\right.$

28. $\left\{\begin{matrix} x+y=10 \\ x^2y=144 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=6 \text{ or } 2\left(1\pm\sqrt{7}\right), \\ y=4 \text{ or } 2\left(4\mp\sqrt{7}\right). \end{matrix}\right.$

29. $\left\{\begin{matrix} x^2+y=11 \\ x+y^2=7 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=3, \\ y=2. \end{matrix}\right.$

30. $\left\{\begin{matrix} x^2-y=21 \\ x+y^2=21 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5 \text{ or } -4, \\ y=4 \text{ or } -5. \end{matrix}\right.$

PAIRS OF EQUATIONS INVOLVING RADICAL QUANTITIES.

85. Solve the following pairs of equations:

1. $\left\{\begin{matrix} \sqrt{x}+\sqrt{y}=6 \\ x+y=20 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=16 \text{ or } 4, \\ y=4 \text{ or } 16. \end{matrix}\right.$

2. $\left\{\begin{matrix} \sqrt{x}+\sqrt{y}=4 \\ x\sqrt{x}+y\sqrt{y}=28 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=9 \text{ or } 1, \\ y=1 \text{ or } 9. \end{matrix}\right.$

3. $\left\{\begin{matrix} x+y=35 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=5 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=27 \text{ or } 8, \\ y=8 \text{ or } 27. \end{matrix}\right.$

4. $\left\{\begin{matrix} x+3\sqrt{x+y}=18-y \\ x^2-y^2=9 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5 \text{ or } 18\frac{1}{8}, \\ y=4 \text{ or } 17\frac{1}{8}. \end{matrix}\right.$

5. $\left\{\begin{matrix} x^2+xy+y^2=a^2 \\ x+(xy)^{\frac{1}{2}}+y=b \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=\dfrac{1}{4b}\left\{a^2+b^2\pm\left(10a^2b^2-3a^4-3b^4\right)^{\frac{1}{2}}\right\}, \\ y=\dfrac{1}{4b}\left\{a^2+b^2\mp\left(10a^2b^2-3a^4-3b^4\right)^{\frac{1}{2}}\right\}. \end{matrix}\right.$

6. $\left\{\begin{array}{l} x^{\frac{3}{2}} + y^{\frac{3}{3}} = 3x \\ x^{\frac{1}{2}} + y^{\frac{1}{3}} = x \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 4,\ 1,\ \text{or } 0. \\ y = 8 \text{ or } 0. \end{array}\right.$

7. $\left\{\begin{array}{l} x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}} = 208 \\ y^2 + x^{\frac{2}{3}}y^{\frac{4}{3}} = 1053 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm\ 8 \text{ or } \pm\frac{8}{5}\sqrt{-65}, \\ y = \pm 27 \text{ or } \pm\frac{27}{5}\sqrt{-65}. \end{array}\right.$

8. $\left\{\begin{array}{l} x^{\frac{2}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 2y^{\frac{1}{3}} = 23 \\ x^{\frac{1}{3}}y^{\frac{1}{3}} = 6 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = 27,\ 8,\ -1,\ \text{or } -216, \\ y = 8,\ 27,\ -216,\ \text{or } -1. \end{array}\right.$

9. $\left\{\begin{array}{l} \frac{x}{y} + 4\sqrt{\frac{x}{y}} = \frac{33}{4} \\ x - y = 5 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 9 \text{ or } \frac{605}{117}, \\ y = 4 \text{ or } \frac{20}{117}. \end{array}\right.$

10. $\left\{\begin{array}{l} y^2 - 8y\sqrt{x} = 64 \\ y - 2\sqrt{xy} = 4 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 2\frac{1}{4}, \\ y = 16. \end{array}\right.$

11. $\left\{\begin{array}{l} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) \\ x - y = 16 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 25, \\ y = 9. \end{array}\right.$

12. $\left\{\begin{array}{l} x^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} = a \\ y^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} = b \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \left\{\frac{a^4}{(a+b)^2}\right\}^{\frac{1}{3}}, \\ y = \left\{\frac{b^4}{(a+b)^2}\right\}^{\frac{1}{3}}. \end{array}\right.$

13. $\left\{\begin{array}{l} x + y + \sqrt{xy} = 185 \\ \sqrt{x^2 + xy} + \sqrt{y^2 + xy} = 175 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 80, \\ y = 45. \end{array}\right.$

GROUPS WITH MORE THAN TWO UNKNOWN QUANTITIES.

86. Solve the following groups of equations:

1. $\left\{\begin{array}{l} xy^2z^3 = 108 \\ yz^2 = 18x \\ 2z = 3x^2y \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \pm 1, \\ y = \pm 2, \\ z = \pm 3. \end{array}\right.$

2. $\left\{\begin{array}{l} xyz = 105 \\ 35x = 3yz \\ 7xy = 15z \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \pm 3, \\ y = \pm 5, \\ z = 7. \end{array}\right.$

3. $\left\{\begin{array}{l} xyz = 231 \\ xyw = 420 \\ xzw = 660 \\ yzw = 1540 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 3, \\ y = 7, \\ z = 11, \\ w = 20. \end{array}\right.$

4. $\left\{\begin{array}{l} xy + 86 = \frac{1}{5}(4x - y)(3x + 2y) \\ xz + 41\frac{1}{2} = \frac{1}{2}(x - z)(x + 3z) \\ yz + 41\frac{1}{7} = \frac{1}{7}(y + 3z)(2y + z) \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \pm 6, \\ y = \pm 1, \\ z = \pm 3. \end{array}\right.$

5. $\left\{\begin{array}{l} x^2 + xy + y^2 = 37 \\ x^2 + xz + z^2 = 28 \\ y^2 + yz + z^2 = 19 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \pm 4 \text{ or } \pm\frac{10}{3}\sqrt{3}, \\ y = \pm 3 \text{ or } \pm\frac{1}{3}\sqrt{3}, \\ z = \pm 2 \text{ or } \mp\frac{8}{3}\sqrt{3}. \end{array}\right.$

6. $\left\{\begin{array}{l} x - y + z = 5 \\ x^2 + y^2 + z^2 = 49 \\ yz = 3x(z - y) \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 3 \text{ or } 2, \\ y = -1 \pm \sqrt{19}, 3, \text{ or } -6, \\ z = \ \ 1 \pm \sqrt{19}, 6, \text{ or } -3. \end{array}\right.$

7. $\left\{\begin{array}{l} x+y+z=13 \\ x^2+y^2+z^2=91 \\ y^2=xz \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=9 \text{ or } 1, \\ y=3, \\ z=1 \text{ or } 9. \end{array}\right.$

8. $\left\{\begin{array}{l} x+y+z=13 \\ x^2+y^2+z^2=61 \\ xy+xz=2yz \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=6 \text{ or } 3, \\ z=3 \text{ or } 6. \end{array}\right.$

9. $\left\{\begin{array}{l} w+x+y+z=21 \\ x+y=2(w+z) \\ 7z^2=36x \\ 1000w+100x+10y+z+4995 \\ \quad =1000z+100y+10x+w \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} w=1, \\ x=7, \\ y=7, \\ z=6. \end{array}\right.$

10. $\left\{\begin{array}{l} x+y=175 \\ z+u=185 \\ x^2+y^2=z^2 \\ xy=uz \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=100, \\ y=75, \\ z=125, \\ u=60. \end{array}\right.$

11. $\left\{\begin{array}{l} x+y+z=23 \\ xy+xz+yz=167 \\ xyz=385 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=7, \\ z=11. \end{array}\right.$

87. PROBLEMS.

1. Find two numbers, such that if the sum of their squares be subtracted from three times their product, the remainder will be 11, and if the difference between their squares be subtracted from twice their product, the remainder will be 14. *Ans.* 5 and 3.

2. Find two numbers whose sum, product, and the difference of whose squares are equal to each other.

Ans. $\frac{3 \pm \sqrt{5}}{2}$ and $\frac{1 \pm \sqrt{5}}{2}$.

3. If a certain number be divided by the product of its two digits, the quotient will be 2, and if 27 be added to the number, the order of the digits will be inverted. What is the number? *Ans.* 36.

4. The sum of three numbers is 33; the difference between the first and second exceeds the difference between the second and third by 6, and the sum of their squares is 441. What are the numbers? *Ans.* 18, 9, and 6.

5. The product of two numbers is 24, and if their sum be added to the sum of their squares, the result will be 62. What are the numbers? *Ans.* 4 and 6.

6. Find two numbers such that, if their product be added to their sum, the result will be 47, and if their sum be subtracted from the sum of their squares, the result will be 62. *Ans.* 7 and 5.

7. The sum of two numbers is 27, and the sum of their cubes is 5103. What are the numbers? *Ans.* 12 and 15.

8. The sum of two numbers is 9, and the sum of their fourth powers is 2417. What are the numbers? *Ans.* 7 and 2.

9. If the product of two numbers be multiplied by the sum of their squares, the result will be 1248, and the difference between their squares is 20. What are the numbers? *Ans.* 6 and 4.

10. Divide the number a into two such parts that the product of their squares shall be b.

Ans. $\frac{a \pm \sqrt{a^2 - 4\sqrt{b}}}{2}$ and $\frac{a \mp \sqrt{a^2 - 4\sqrt{b}}}{2}$.

11. Divide the number 20 into two such parts that the product of their squares shall be 9216. *Ans.* 12 and 8.

12. Find two numbers, such that their product shall be equal to the difference between their squares, and the sum of their squares shall be equal to the difference between their cubes.

Ans. $\frac{5 \pm \sqrt{5}}{4}$ and $\pm \frac{\sqrt{5}}{2}$.

13. A certain number consists of two digits. The left-hand digit is equal to three times the right-hand digit; and if 12 be subtracted from the number, the remainder will be equal to the square of the left-hand digit. What is the number? *Ans.* 93.

14. Find two numbers, such that if their difference be multiplied by the greater, the product will be 40, and if the difference be multiplied by the less, the product will be 15.

Ans. 8 and 3.

15. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in passing over 120 yards; but if the circumference of each wheel be increased by one yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each wheel.

Ans. Fore-wheel, 4 yards; hind-wheel, 5 yards.

16. Find two numbers such that, if the less be subtracted from three times the greater, the remainder will be 35, and if four times the greater be divided by three times the less plus one, the quotient will be equal to the less.

Ans. 13 and 4.

17. Find two numbers whose product is 128, and the difference between whose squares is 192. *Ans.* 16 and 18.

18. A certain number consists of two digits. If 9 be added to the number, the order of the digits will be inverted, and if 10 be subtracted from the number, the remainder will be equal to the sum of the squares of the digits. What is the number? *Ans.* 23.

19. A certain number consists of two digits. The number is equal to three times the sum of its digits, and three times the number is equal to the square of the sum of its digits. What is the number? *Ans.* 27.

20. Find two numbers whose product is 6 times their sum, and the sum of whose squares is 325.

Ans. 10 and 15.

21. Find two numbers whose sum is 6, and the difference between whose third powers is 56. *Ans.* 4 and 2.

22. Find two numbers such that, if three times the square of the greater be added to twice the square of the less, the result will be 110, and if half their product be added to the square of the less, the result will be 4.

Ans. 6 and 1.

23. Two persons, A and B, leave Columbia, and walk in the same direction at uniform rates. B starts 2 hours after A, and, after traveling 30 miles, overtakes him; but had each traveled half a mile more per hour, B would have overtaken A 42 miles from Columbia. At what rate did each travel? *Ans.* A, $2\frac{1}{2}$ miles per hour; B, 3 miles per hour.

24. The sum of A and the square root of B is 21, and if the square root of A be subtracted from B, the remainder will be 21. Find the values of A and B.

Ans. $A = 16$, $B = 25$.

25. Find two numbers, the sum of whose squares is 58, and the sum of whose cubes is 370. *Ans.* 7 and 3.

26. The year in which an important event in American History occurred is expressed by four digits. The sum of the digits is 21; the sum of the two middle digits is twice the sum of the other two; the square of the fourth digit, counting from the left, is $5\frac{1}{4}$ times the second digit; and if 4995 be added to the number of the year, the order of the digits will be inverted. In what year did the event occur?

Ans. 1776.

27. Two men, A and B, sell a certain number of mules for \$568. B sells four more mules than A. B would have received \$200 for the number A sold, and A would have received \$336 for the number B sold. Find the number sold by each, and the rates at which they sold.

Ans. A sold 4 mules at \$42 each, and B sold 8 mules at \$50 each.

28. A poulterer going to market to buy turkeys met with four flocks. In the second there were 6 more than three times the square root of double the number in the first; the third contained three times as many as the first and second together; the fourth contained 6 more than the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

Ans. 18, 24, 126, 1770.

29. There were two shelves of books, the upper one containing one more book than the lower one. A certain number having been taken from the upper shelf, and as many as then remained from the lower shelf, it was found that if the square of the number remaining on the lower shelf be added

to the square root of that number, the result would be equal to the quotient obtained by dividing 72 by the excess of the number taken from the upper shelf above unity. How many books were taken from the upper shelf? *Ans.* 5.

30. A and B were going to market, the first with cucumbers, and the second with three times as many eggs. If B had given all his eggs for the cucumbers, A would have lost 10 cents, according to the rate at which they were then selling. If A had given three-fifths of his cucumbers for all of B's eggs, B would have *lost* 6 cents, according to the same rate, but would have *gained* upon the whole the price of six eggs, if he could have sold the cucumbers at 6 cents each. Find the number of eggs and cucumbers, and the price of each.

Ans. 30 eggs and 10 cucumbers; eggs, 1 cent each; cucumbers, 4 cents each.

31. A person bought a certain number of larks and sparrows for $.72. He paid as many cents per dozen for larks as there were sparrows, and as many cents per score for sparrows as there were larks. If he had bought 10 more of each (the price of larks remaining the same), and had given as much per dozen for sparrows as he gave per score for larks, he would have paid for all $3.05. Find the number of each.

Ans. 15 larks and 36 sparrows.

RATIO.

88.—1. Which is greater, $16:15$ or $17:14$?

Ans. $17:14$.

2. Which is greater, $a+2:\frac{a}{2}+4$ or $a+4:\frac{a}{2}+5$?

Ans. $a+4:\frac{a}{2}+5$.

3. Find the inverse ratio of D^2 to Q. *Ans.* $\frac{Q}{D^2}$.

4. Find the ratio which is compounded of the subduplicate ratio of x^2 to y^2 and the duplicate ratio of $\sqrt{x}$ to $\sqrt{y}$.

Ans. $\frac{x^2}{y^2}$.

5. Show that the ratio of a^2-x^2 to a^2+x^2 is greater than the ratio of $a-x$ to $a+x$, if x is not equal to a.

6. Show that the ratio compounded of $x+y:a$, $x-y:b$, and $b:\frac{x^2-y^2}{a}$, is a ratio of equality.

7. How is the ratio of a to $a-2b$ affected by adding d to both terms. *Ans.* The ratio is diminished.

8. Two vessels, A and B, each contain a mixture of wine and water, A in the ratio of 3 to 2, and B in the ratio of 7 to 3. How many gallons must be drawn from each, in order to make a third mixture which shall contain 5 gallons of water and 11 gallons of wine?

Ans. 2 gallons from A, and 14 gallons from B.

PROPORTION.

89.—1. Find a mean proportional between to a^2 and b^2.

Ans. ab.

2. Find a fourth proportional to a, b, and c. *Ans.* $\frac{bc}{a}$.

3. Find a third proportional to a and b. *Ans.* $\frac{b^2}{a}$.

4. If $a:b=c:d$, show that $(a+mb)^2:(c+md)^2=a^2-b^2:c^2-d^2$.

5. If $a:b=c:d$, and $c:e=e:d$, show that $a:b=c^3:e^3$.

6. If $x:y=a^2:b^2$, and $a:b=\sqrt{a+x}:\sqrt{a-y}$, show that $2x:a=x-y:y$.

7. Solve the equation $6x+a:4x+b=3x-b:2x-a$.

Ans. $x=\frac{b^2-a^2}{4a-b}$.

8. Solve the equations

$$\left\{\begin{matrix} x^3-y^3:(x-y)^3=61:1 \\ x:8=40:y \end{matrix}\right\}.$$

Ans. $\left\{\begin{matrix} x=\pm 20 \text{ or } \pm 16, \\ y=\pm 16 \text{ or } \pm 20. \end{matrix}\right.$

9. Solve the equations

$$\left\{\begin{matrix} \sqrt{y}-\sqrt{y-x}=\sqrt{20-x} \\ \sqrt{y-x}:\sqrt{20-x}=3:2 \end{matrix}\right\}.$$

Ans. $\left\{\begin{matrix} x=16, \\ y=25. \end{matrix}\right.$

10. Solve the equations

$$\left\{\begin{matrix} x+y:x=7:5 \\ xy+y^2=126 \end{matrix}\right\}.$$

Ans. $\left\{\begin{matrix} x=\pm 15, \\ y=\pm 6. \end{matrix}\right.$

11. Solve the equations

$$\left\{\begin{matrix} 2a : 1 = x^2 + y^2 : a \\ n : m = x - y : x + y \end{matrix}\right\}. \quad Ans. \quad \left\{\begin{matrix} x = \pm \dfrac{a(m+n)}{\sqrt{m^2+n^2}}, \\ y = \pm \dfrac{a(m-n)}{\sqrt{m^2+n^2}}, \end{matrix}\right.$$

12. What number must be added to a and subtracted from b, that the sum may be to the difference as m is to n?

$$Ans. \ \frac{bm - an}{m+n}.$$

13. Divide the number 100 into two such parts, that 6 times their product shall be to the sum of their squares as 24 is to 17. *Ans.* 80 and 20.

14. Divide the number 18 into two such parts, that their squares may be in the ratio of 25 to 16.

Ans. 10 and 8.

15. Divide the number 14 into two such parts, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 16 is to 9.

Ans. 8 and 6.

16. The captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings, it was found that the privateer had run at the rate of 10 knots per hour, and the trading vessel at the rate of 8 knots per hour. Find the distance sailed by the privateer while in pursuit. *Ans.* 25 miles.

VARIATION.

90.—1. $y \propto x^2$, and when $x = 3$, $y = 27$. Find the equation expressing the relation between y and x.

Ans. $y = 3x^2$.

2. If $y^2 \propto a^2 - x^2$, and $y = \frac{b^2}{a}$ when $x = \sqrt{a^2 - b^2}$, what is the equation expressing the relation between y and x?

Ans. $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

3. If $x + y \propto x - y$, show that $x^2 + y^2 \propto xy$.

4. If $z \propto x + my$, and $y = 2$ and $z = 3$ when $x = 1$, and $y = 3$ and $z = 4$ when $x = 2$, what is the value of m? *Ans.* $m = -2$.

5. The time in which a pendulum makes one vibration varies as the square root of its length, and the length of a pendulum which vibrates once in a second is 39.2 inches. Find the length of one which vibrates 56 times in a minute.

Ans. 45 inches.

PERMUTATIONS.

91.—1. Find the number of different permutations of 9 things, taken 4 at a time. *Ans.* 3024.

2. For how many days can 5 persons be placed in different orders of arrangement around a table at dinner?

Ans. 120.

3. The number of permutations of n things, taken 3 at a time, is to the number, taken 5 at a time, as 1 is to 12. Find the value of n. *Ans.* $n = 7$.

4. The number of things is to the number of permutations of the things, taken 3 at a time, as 1 is to 20. How many things are there? *Ans.* 3.

5. The number of permutations of n things, taken 3 at a time, is one-fifth of the number of permutations of $n + 2$ things, taken 3 at a time. Find the value of n.

Ans. $n = 4$.

6. How many different permutations may be made of the letters in the word *Calcutta*, taken all together?

Ans. 5040.

COMBINATIONS.

92.—1. Out of 100 soldiers, how many different bodies of 4 sentinels each can be chosen? *Ans.* 3921225.

2. The number of combinations of n things, taken 3 at a time, is to the number, taken 5 at a time, as 5 is to 18. Find the value of n. *Ans.* $n = 12$.

3. If the number of permutations of n things, taken 3 at a time, is equal to the number of combinations of n things, taken 4 at a time, what is the value of n?

Ans. $n = 7$.

4. Of 9 things, how many must be taken at a time, in order that the number of combinations may be the greatest possible? *Ans.* 5 or 4.

5. A person wishes to make up as many parties as he can out of 20 friends. How many should he invite at a time?

Ans. 10.

THE BINOMIAL FORMULA.

93.—1. Expand $(a + x)^4$.

Ans. $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.

2. Expand $(2x + 3y)^5$.

Ans. $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$.

3. Expand $(a^2 - x^2)^4$.

Ans. $a^8 - 4a^6x^2 + 6a^4x^4 - 4a^2x^6 + x^8$.

4. Expand $\left(1 + \frac{x}{2}\right)^5$.

Ans. $1 + \frac{5x}{2} + \frac{5x^2}{2} + \frac{5x^3}{4} + \frac{5x^4}{16} + \frac{x^5}{32}$.

5. Expand $\left(\frac{1}{2}x - 2y\right)^7$.

Ans. $\frac{x^7}{128} - \frac{7}{32}x^6y + \frac{21}{8}x^5y^2 - \frac{35}{2}x^4y^3 + 70x^3y^4 - 168x^2y^5 + 224xy^6 - 128y^7$.

6. Expand $(a^2 - ax)^{10}$.

Ans. $a^{20} - 10a^{19}x + 45a^{18}x^2 - 120a^{17}x^3 + 210a^{16}x^4 - 252a^{15}x^5 + 210a^{14}x^6 - 120a^{13}x^7 + 45a^{12}x^8 - 10a^{11}x^9 + a^{10}x^{10}$.

7. Expand $(\sqrt[3]{x} - \sqrt[3]{y})^6$.

Ans. $x^2 - 6\sqrt[3]{x^5y} + 15\sqrt[3]{x^4y^2} - 20xy + 15\sqrt[3]{x^2y^4} - 6\sqrt[3]{xy^5} + y^2$.

8. Expand $(1 + x)^{\frac{1}{2}}$.

Ans. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots$.

9. Expand $(1-x)^{-\frac{1}{2}}$.

Ans. $1+\frac{x}{2}+\frac{3x^2}{8}+\frac{5x^3}{16}+\frac{35x^4}{128}+\ldots.$

10. Expand $\frac{1}{(1-x)^3}$.

Ans. $1+3x+6x^2+10x^3+15x^4+\ldots.$

11. Show that

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=1+\frac{x}{a+x}+\frac{3x^2}{2(a+x)^2}+\frac{5x^3}{2(a+x)^3}+\ldots.$$

12. Expand $(1+x)^{\frac{1}{\sqrt{2}}}$.

Ans. $1+\frac{x}{\sqrt{2}}+\frac{x^2(1-\sqrt{2})}{2}+\frac{x^3(1-\sqrt{2})(1-2\sqrt{2})}{12\sqrt{2}}+\ldots.$

13. Find the 8th term of the expansion of $(1+x)^{11}$.

Ans. $330x^7$.

14. Find the 6th term of the expansion of $(a^2-2ab)^6$.

Ans. $-192a^7b^5$.

15. Find the 4th term of the expansion of $(x^3+b^2)^5$.

Ans. $10x^4b^6$.

16. Find the 6th term of the expansion of $\left(\frac{\sqrt[n]{x}}{a}+\frac{\sqrt[n]{y}}{b}\right)^{12}$.

Ans. $\frac{792x^{\frac{7}{n}}y^{\frac{5}{n}}}{a^7b^5}$.

THE HIGHER ROOTS OF QUANTITIES.

94.—1. Find the fourth root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$. *Ans.* $a + x$.

2. Find the fifth root of $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$. *Ans.* $2x + 3y$.

3. Find the fourth root of $a^8 - 4a^6x^2 + 6a^4x^4 - 4a^2x^6 + x^8$. *Ans.* $a^2 - x^2$.

4. Find the fifth root of $1 + \frac{5x}{2} + \frac{5x^2}{2} + \frac{5x^3}{4} + \frac{5x^4}{16} + \frac{x^5}{32}$. *Ans.* $1 + \frac{x}{2}$.

5. Find the tenth root of $a^{20} - 10a^{19}x + 45a^{18}x^2 - 120a^{17}x^3 + 210a^{16}x^4 - 252a^{15}x^5 + 210a^{14}x^6 - 120a^{13}x^7 + 45a^{12}x^8 - 10a^{11}x^9 + a^{10}x^{10}$. *Ans.* $a^2 - ax$.

6. Find the sixth root of $x^2 - 6x^{\frac{5}{3}}y^{\frac{1}{3}} + 15x^{\frac{4}{3}}y^{\frac{2}{3}} - 20xy + 15x^{\frac{2}{3}}y^{\frac{4}{3}} - 6x^{\frac{1}{3}}y^{\frac{5}{3}} + y^2$.

DECOMPOSITION OF RATIONAL FRACTIONS.

95. Separate each of the following fractions into its partial fractions:

1. $\frac{2x - 3}{x^2 - 3x + 2}$. *Ans.* $\frac{1}{x - 1} + \frac{1}{x - 2}$.

2. $\frac{6x^2 - 4x - 6}{(x-1)(x-2)(x-3)}$. *Ans.* $\frac{-2}{x-1} - \frac{10}{x-2} + \frac{18}{x-3}$.

3. $\dfrac{x^2}{(x^2-1)(x-2)}$. *Ans.* $\dfrac{\frac{1}{6}}{x+1} - \dfrac{\frac{1}{2}}{x-1} + \dfrac{\frac{4}{3}}{x-2}$.

4. $\dfrac{9+34x+29x^2}{(1+x)(1+2x)(1+3x)}$.

Ans. $\dfrac{2}{1+x} + \dfrac{3}{1+2x} + \dfrac{4}{1+3x}$.

5. $\dfrac{1}{x^4-a^4}$. *Ans.* $\dfrac{\frac{1}{4a^3}}{x-a} - \dfrac{\frac{1}{4a^3}}{x+a} - \dfrac{\frac{1}{2a^2}}{x^2+a^2}$.

6. $\dfrac{x^2+hx+k}{(x-a)(x-b)(x-c)}$.

Ans. $\dfrac{\frac{a^2+ah+k}{(a-b)(a-c)}}{x-a} + \dfrac{\frac{b^2+bh+k}{(b-a)(b-c)}}{x-b} + \dfrac{\frac{c^2+ch+k}{(c-a)(c-b)}}{x-c}$.

SERIES.

ARITHMETICAL PROGRESSION.

96.—1. Find the 15th term of the series 3, 7, 11, ...

Ans. 59.

2. Find the 11th term of the series 5, 1, −3, ...

Ans. −35.

3. Find the 20th term of the series 57, 54, 51, ...

Ans. 0.

4. Sum to 20 terms the series 1, 3, 5, ...

Ans. 400.

5. Sum to n terms the series $\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \ldots$.

Ans. $\frac{n-1}{2}$.

6. Form an A. P. of 5 terms whose extremes shall be 117 and 477. *Ans.* 117, 207, 297, 387, 477.

7. Given $a=1$, $s=280$, and $n=32$, to find d and l.

Ans. $\begin{cases} d=\frac{1}{2}, \\ l=16\frac{1}{2}. \end{cases}$

8. Given $a=15$, $d=-2$, and $s=60$, to find l and n.

Ans. $\begin{cases} l=5 \text{ or } -3. \\ n=6 \text{ or } 10. \end{cases}$

9. Find the sum of the first n terms of the progression 1, 2, 3, 4, ...

Ans. $s=\frac{n}{2}(1+n)$.

10. The n^{th} term of an A. P. is $\frac{3n-1}{6}$: find the first term, common difference, and the sum of the first n terms.

Ans. $\begin{cases} a=\frac{1}{3}, \\ d=\frac{1}{2}, \\ s=\frac{n}{12}(3n+1). \end{cases}$

11. A sets out from a place and travels 1 mile the first day, 2 the second, 3 the third, and so on. Six days later B sets out from the same place and travels 15 miles a day in the same direction as A. How long will A travel before he is overtaken by B? *Ans.* 9 or 20 days.

12. A sets out from a place and travels 1 mile the first day, 2 the second, 3 the third, and so on. B sets out m days later from the same place and travels n miles a day in the same direction as A. How long will A travel before he is overtaken by B? *Ans.* $\frac{2n-1 \pm \sqrt{(2n-1)^2 - 8mn}}{2}$ days.

Show that B will *never* overtake A if $m > \frac{(2n-1)^2}{8n}$.

13. The distance from P to Q is 168 miles. A sets out from P toward Q and travels 3 miles the first day, 5 the second, 7 the third, and so on. At the same time B sets out from Q toward P, and travels 4 miles the first day, 6 the second, 8 the third, and so on. When will they meet?

Ans. At the end of 8 days.

14. A certain number consists of three digits, which are in arithmetical progression. If the number be divided by the sum of its digits the quotient will be 26; and if 198 be added to the number the order of its digits will be inverted. What is the number? *Ans.* 234.

15. Find three numbers in A. P. whose sum is 18, and the sum of whose squares is 158. *Ans.* 1, 6, 11.

16. Find four numbers in A. P. whose sum is 16, and the sum of whose squares is 69. *Ans.* $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, $5\frac{1}{2}$.

17. Find three numbers in A. P., the sum of whose squares is 308, and the square of whose arithmetical mean exceeds the product of the extremes by 4. *Ans.* 8, 10, 12.

18. Find four numbers in A. P. whose sum is 56, and whose continued product is 9360. *Ans.* 2, 10, 18, 26.

GEOMETRICAL PROGRESSION.

97.—1. Find the sum of the first ten terms of the series $1, \frac{2}{3}, \frac{4}{9}, \ldots$. *Ans.* $\frac{174075}{59049}$.

2. Form a G. P. of 9 terms whose extremes shall be 3 and 768. *Ans.* 3, 6, 12, 24, 48, 96, 192, 384, 768.

3. Find the 7th term of the series $-21, 14, -9\frac{1}{3}, \ldots$. *Ans.* $-\frac{448}{243}$.

4. The arithmetical mean between x and y is double the geometrical mean; find the value of $\frac{x}{y}$. *Ans.* $7 \pm 4\sqrt{3}$.

5. Multiply $1 + \frac{1}{2} + \frac{1}{4} + \ldots$. to infinity by $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$. to infinity. *Ans.* $\frac{4}{3}$.

6. Find the value of x in the equation

$$x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{8} + \ldots \text{ to infinity} = \frac{2}{3}.$$

Ans. $x = 1$.

7. Find the ratio of an infinite decreasing G. P., of which the first term is a, and the sum of the terms b.

Ans. $r = \frac{b - a}{b}$.

8. Find the ratio of an infinite decreasing G. P., in which each term is m times the sum of all the terms which follow it.

Ans. $r = \frac{1}{m + 1}$.

9. Find the sum of the first n terms of a G. P. whose first term is a, and fourth term d.

$$Ans.\ s = \frac{\sqrt[3]{\frac{d^n}{a^{n-3}}} - a}{\sqrt[3]{\frac{d}{a}} - 1}.$$

10. The extremes of a G. P. of 6 terms are 2 and 8; find the product of all the terms. *Ans.* 4096.

11. The sum of four numbers in G. P. is 30, and the quotient obtained by dividing the last term by the sum of the means is $1\frac{1}{3}$; find the numbers. *Ans.* 2, 4, 8, 16.

12. The sum of the first and third of four numbers in G. P. is 148, and the sum of the second and fourth is 888; find the numbers. *Ans.* 4, 24, 144, 864.

13. Find the geometrical mean of a G. P. whose extremes are 2 and 32. *Ans.* 8.

14. The continued product of three numbers in G. P. is 8, and the sum of their cubes is 73; find the numbers.

Ans. 1, 2, 4.

15. If P be the continued product of n quantities in G. P., s their sum, and s_1 the sum of their reciprocals, show that $P^2 = \left(\frac{s}{s_1}\right)^n$.

THE DIFFERENTIAL METHOD.

98.—1. Find the 12th term of the series 1, 5, 15, 35, 70, 126, ... *Ans.* 1365.

2. Find the 15th term of the series 1, 4, 10, 20, 35, ...

Ans. 670.

3. Find the 20th term of the series 6, 10, 15, 21, ...

Ans. 253.

4. Find the n^{th} term of the series 2, 6, 12, 20, 30, ...

Ans. $n^2 + n$.

5. Find the 50th term of the series 1, 3, 6, 10, 15, ...

Ans. 1275.

6. Find the n^{th} term of the series 6, 24, 60, 120, 210, ...

Ans. $n(n+1)(n+2)$.

7. Find the sum of 10 terms of the series 3, 5, 7, 9, 11, ...

Ans. 120.

8. Find the sum of n terms of the series 1, 3, 6, 10, 15, ...

Ans. $\frac{n(n+1)(n+2)}{6}$.

9. Find the sum of n terms of the series 2, 6, 12, 20, 30, ...

Ans. $\frac{n(n+1)(n+2)}{3}$.

10. Find the sum of n terms of the series 1^3, 2^3, 3^3, 4^3, ...

Ans. $\left(\frac{n^2+n}{2}\right)^2$.

11. Find the sum of n terms of the series 1, 4, 9, 16, 25, 36, ...

Ans. $\frac{(2n^2+3n+1)n}{6}$.

12. Show that the n^{th} term of the series 6, 24, 60, 120, 210, ... is 3 times the sum of n terms of the series 2, 6, 12, 20, 30, ...

INTERPOLATION.

99.—1. Find the 5th term of the series of which the 6th differences vanish and the 1st, 2d, 3d, 4th, 6th, 7th terms are 11, 18, 30, 50, 132, 209. *Ans.* 82.

2. Find the 2d term of the series of which the 4th differences vanish, and the 1st, 3d, 4th, 5th terms are 3, 15, 30, 55. *Ans.* 7.

3. Given the square roots of 19, 20, 21, 23 to find that of 22.

4. Insert three equidistant terms between every two consecutive terms of the series 1, 4, 10, 20, 35.

Ans. $1, \frac{195}{128}, \frac{280}{128}, \frac{385}{128}, 4, \frac{663}{128}, \ldots$.

DEVELOPMENT OF EXPRESSIONS INTO SERIES.

100. Convert each of the following expressions into an infinite series:

1. $\frac{a}{a+x}$. *Ans.* $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \ldots$.

2. $\frac{a}{a-x}$. *Ans.* $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \ldots$.

3. $\frac{1+2x}{1-x-x^2}$.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+ \ldots$.

4. $\frac{1+2x}{1-x+x^2}$.

Ans. $1+3x+2x^2-x^3-3x^4-2x^5+x^6+ \ldots$.

5. $\frac{x}{(1-x)^2}$. *Ans.* $x+2x^2+3x^3+4x^4+5x^5+\ldots$.

6. $\frac{1}{3x-x^2}$. *Ans.* $\frac{1}{3x}+\frac{1}{9}+\frac{x}{27}+\frac{x^2}{81}+\ldots$.

7. $\sqrt{a-x}$.
Ans. $\sqrt{a}\left(1-\frac{x}{2a}-\frac{x^2}{2\cdot 4a^2}-\frac{3x^3}{2\cdot 4\cdot 6a^3}-\ldots\right)$.

8. $\sqrt[3]{1-x}$. *Ans.* $1-\frac{x}{3}-\frac{2x^2}{3\cdot 6}-\frac{2\cdot 5x^3}{3\cdot 6\cdot 9}-\ldots$.

9. $(a-b)^{\frac{2}{3}}$.
Ans. $a^{\frac{2}{3}}\left(1-\frac{2b}{3a}-\frac{2b^2}{3\cdot 6a^2}-\frac{2\cdot 4b^3}{3\cdot 6\cdot 9a^3}-\ldots\right)$.

10. $x(1-x)^{-\frac{1}{5}}$.
Ans. $x+\frac{x^2}{5}+\frac{6x^3}{2\cdot 5^2}+\frac{6\cdot 11x^4}{2\cdot 3\cdot 5^3}+\ldots$.

RECURRING SERIES.

101. Find the generating fraction of each of the following series:

1. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\ldots$. *Ans.* $\frac{a}{a+x}$.

2. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\ldots$. *Ans.* $\frac{a}{a-x}$.

3. $1+3x+4x^2+7x^3+11x^4+18x^5+\ldots$.
Ans. $\frac{1+2x}{1-x-x^2}$.

4. $1 + 3x + 2x^2 - x^3 - 3x^4 - 2x^5 + x^6 + \ldots$.

Ans. $\dfrac{1 + 2x}{1 - x + x^2}$.

5. $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots$. *Ans.* $\dfrac{x}{(1 - x)^2}$.

6. $\dfrac{1}{3x} + \dfrac{1}{9} + \dfrac{x}{27} + \dfrac{x^2}{81} + \ldots$. *Ans.* $\dfrac{1}{3x - x^2}$.

7. $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5 + \ldots$.

Ans. $\dfrac{(1 + x)^2 - 2x^2}{(1 - x)^2 - 3x^2}$.

8. $1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \ldots$.

Ans. $\dfrac{1}{1 - x + x^2}$.

9. $1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - \ldots$.

Ans. $\dfrac{1}{1 + x + x^2}$.

10. $x + 3x^3 + 6x^3 + 10x^4 + \ldots$. *Ans.* $\dfrac{x}{(1 - x)^3}$.

REVERSION OF SERIES.

102.—Revert the series in the following equations:

1. $y = x + x^2 + x^3 + \ldots$.

Ans. $x = y - y^2 + y^3 - y^4 + y^5 - \ldots$.

2. $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \ldots$.

Ans. $y = x + \dfrac{x^2}{1 \cdot 2} + \dfrac{x^3}{1 \cdot 2 \cdot 3} + \dfrac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$.

3. $y = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \ldots$.

Ans. $x = y - 3y^2 + 13y^3 - 67y^4 + \ldots$.

4. $y = 1 - 2x + 3x^2$.

Ans. $x = -\frac{1}{2}(y-1) + \frac{3}{8}(y-1)^2 - \frac{9}{16}(y-1)^3 + \frac{135}{128}(y-1)^4 - \ldots$.

5. $y = a + bx + cx^2$.

Ans. $x = \frac{1}{b}(y-a) - \frac{c}{b^3}(y-a)^2 + \frac{2c^2}{b^5}(y-a)^3 - \frac{5c^3}{b^7}(y-a)^4 + \ldots$.

6. $y = 1 + x - 2x^2 + x^3$.

Ans. $x = y - 1 + 2(y-1)^2 + 7(y-1)^3 + 30(y-1)^4 + \ldots$.

7. $y = a + bx + cx^2 + dx^3$.

Ans. $x = \frac{1}{b}(y-a) - \frac{c}{b^3}(y-a)^2 + \frac{2c^2 - bd}{b^5}(y-a)^3 - \frac{5c^3 - 5bcd}{b^7}(y-a)^4 + \ldots$.

8. $y = 1 + 2x + 4x^2 + 8x^3 + \ldots$.

Ans. $x = \frac{1}{2}(y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{2}(y-1)^3 - \frac{1}{2}(y-1)^4 + \ldots$.

9. $y = x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \ldots.$

Ans. $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$

10. $y = ax + bx^2 + cx^3 + \ldots.$

Ans. $x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4 + \ldots.$

LOGARITHMS AND EXPONENTIAL EQUATIONS.

LOGARITHMS.

103.—1. Show that $\log_{\frac{1}{2}} 4 = -2$.

2. Show that a negative number has no real logarithm.

3. Show that $\log_a 0 = \infty$ or $+\infty$, according as $a > 1$ or < 1.

4. Show that $\log(a^2 - b^2) - \log(a - b) = \log(a + b)$.

5. Show that $\log 2 - \log(\sqrt{5} - \sqrt{2}) = \log 2 + \log(\sqrt{5} + \sqrt{2}) - \log 3$.

6. In a G. P. $a = \frac{1}{4}$, $l = 64$, and $r = 4$; what is the value of n? *Ans.* $n = 5$.

7. In a G. P. $a = 1$, $l = 32$, and $s = 63$; what is the value of n? *Ans.* $n = 6$.

8. In a G. P. $a = 1$, $r = 3$, and $s = 121$; what is the value of n? *Ans.* $n = 5$.

9. In a G. P. $l = 0$, $r = \frac{1}{2}$, and $s = 2$; what is the value of n? *Ans.* $n = \infty$.

10. Given $a^1 \times a^3 \times a^5 \times a^7 \ldots = p$ to find the number of factors $a^1, a^3, a^5, a^7, \ldots$. *Ans.* $\sqrt{\frac{\log p}{\log a}}$.

EXPONENTIAL EQUATIONS.

104.—1. Solve $20^x = 100$. *Ans.* $x = 1.537$.

2. Solve $ab^{\frac{1}{x}} = n$. *Ans.* $x = \frac{\log b}{\log n - \log a}$.

3. Solve $a^{2x} - a^x = 1$. *Ans.* $x = \frac{\log\left[\frac{1}{2}(1 \pm \sqrt{5})\right]}{\log a}$.

4. Solve $\left\{\begin{matrix} x^{\sqrt[4]{x}+\sqrt[4]{y}} = y^{\frac{8}{3}} \\ y^{\sqrt[4]{x}+\sqrt[4]{y}} = x^{\frac{8}{3}} \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = \left(-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\right)^8, \\ y = \left(-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\right)^4. \end{matrix}\right.$

5. Solve $(2^3)^x (3^2)^x = 4.9$. *Ans.* $x = .37166$.

6. Solve $a^{bx+d} = c$. *Ans.* $x = \frac{\log c - d \log a}{b \log a}$.

7. Solve $a^{mx} b^{nx} = c$. *Ans.* $x = \frac{\log c}{m \log a + n \log b}$.

8. Solve $3^{2x} \times 5^{3x-4} = 7^{x-1} \times .11^{2-x}$. *Ans.* $x = 1.242$.

9. Solve $\left\{\begin{matrix} 14^x = 63y \\ 17^x = 87y \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = 1.6624, \\ y = 1.2764. \end{matrix}\right.$

10. Solve $\left\{\begin{matrix} 2^x \times 3^y = 560 \\ 5x = 7y \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = 4.2818, \\ y = 3.0584. \end{matrix}\right.$

11. Solve $\left\{\begin{matrix} x^y = y^x \\ x^3 = y^2 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = 2.25, \\ y = 3.375. \end{matrix}\right.$

COMPOUND INTEREST AND ANNUITIES.

COMPOUND INTEREST.

105.—1. What time is required for any sum of money to double itself at three per cent. compound interest?

Ans. 23.45 years.

2. A bottle of wine that originally cost 20 cents was put away 200 years ago; what would it be worth now, allowing 5 per cent. compound interest? *Ans.* $3458.10.

3. What will be the amount of $5000, at 4 per cent. compound interest, for 40 years? *Ans.* $24005.10.

4. In what time will $5 amount to $9, at 5 per cent. compound interest? *Ans.* 12.04 years.

5. A capital of $1000 in 6 years, at compound interest, amounted to $1800; what was the rate per cent.?

Ans. 10.3.

6. What principal, at 4 per cent compound interest for 4 years, will amount to $350.95\frac{1}{4}$? *Ans.* $300.

ANNUITIES.

106.—1. What is the amount of an annuity of $50 for 6 years, at 6 per cent. per annum, the interest being compounded every year? *Ans.* $348.56.

2. In what time will an annuity of \$20 amount to \$1000, at 4 per cent. compound interest? *Ans.* 28 years.

3. What is the present value of an annuity of \$50 for 20 years, at $3\frac{1}{2}$ per cent. per annum, the interest being compounded every year? *Ans.* \$710.62.

4. What is the present value of a perpetual annuity of \$3000, at 3 per cent. per annum, the interest being compounded every year? *Ans.* \$100000.

THEORY OF EQUATIONS.

GENERAL PROPERTIES.

107.—1. If 3 is a root of the equation $x^3 - 4x^2 + x + c = 0$, what is the value of c? *Ans.* $c = 6$.

2. Show that 2 is a root of the equation $x^3 - x^2 - 14x + 24 = 0$.

3. Show that -3 is a root of the equation $x^3 - 37x - 84 = 0$.

4. Show that 5 and -1 are roots of the equation $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$.

5. Two roots of the equation $x^4 + x^3 - 19x^2 + 11x + 30 = 0$ are -1 and 2; what are the other roots?

Ans. 3, -5.

6. Find the equation whose roots are 0, -1, 2, -5.

Ans. $x^4 + 4x^3 - 7x^2 - 10x = 0$.

7. Find the equation whose roots are $4\sqrt{3}$, $-4\sqrt{3}$, $5 \pm 2\sqrt{-1}$. *Ans.* $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$.

8. One root of the equation $x^3 - 11x^2 + 37x - 35 = 0$ is $3 + \sqrt{2}$; what are the other roots?

Ans. 5 and $3 - \sqrt{2}$.

9. One root of the equation $x^4 - 3x^3 - 42x - 40 = 0$ is $-\frac{1}{2}(3 + \sqrt{-31})$; what are the other roots?

Ans. 4, -1, $-\frac{1}{2}(3 - \sqrt{-31})$.

10. Has the equation $x^3 - x^2 - 8x + 12 = 0$ a real root? Why?

11. Has the equation $x^4 + 14x^3 + 61x^2 - 84x - 36 = 0$ any real roots? Why?

TRANSFORMATION OF EQUATIONS.

108. Transform each of the following equations into another whose roots shall be the negatives of those of the given equation:

1. $x^3 - 7x^2 + 13x - 3 = 0$. *Ans.* $x^3 + 7x^2 + 13x + 3 = 0$.
2. $x^4 - 3x^3 + 3x^2 + 17x - 18 = 0$.
   Ans. $x^4 + 3x^3 + 3x^2 - 17x - 18 = 0$.
3. $x^4 + 2x^3 - 7x - 1 = 0$. *Ans.* $x^4 - 2x^3 + 7x - 1 = 0$.
4. $x^3 - 1 = 0$. *Ans.* $x^3 + 1 = 0$.
5. $x^4 + 5x^2 + 5 = 0$. *Ans.* $x^4 + 5x^2 + 5 = 0$.

109. Transform each of the following equations into another in which the coefficients shall be entire, that of the first term being unity:

1. $x^3 + 2x^2 + \frac{x}{4} + \frac{1}{9} = 0$. *Ans.* $y^3 + 12y^2 + 9y + 24 = 0$.

2. $x^3 - \frac{7x^2}{3} + \frac{11x}{36} - \frac{25}{72} = 0.$

Ans. $y^3 - 14y^2 + 11y - 75 = 0.$

3. $x^4 - \frac{5x^3}{6} + \frac{5x^2}{12} - \frac{7x}{150} - \frac{13}{900} = 0.$

Ans. $y^4 - 25y^3 + 375y^2 - 1260y - 11700 = 0.$

4. $x^4 - \frac{4}{3}x^3 - \frac{3}{8}x + \frac{5}{72} = 0.$

Ans. $y^4 - 48y^3 - 81y + 90 = 0.$

110.—1. Find an equation whose roots are less by 3 than those of the equation $x^3 - 27x - 36 = 0.$

Ans. $y^3 + 9y^2 - 90 = 0.$

2. Find an equation whose roots are less by 5 than those of the equation $x^4 - 18x^3 - 32x^2 + 17x + 9 = 0.$

Ans. $y^4 + 2y^3 - 152y^2 - 1153y - 2331 = 0.$

3. Find an equation whose roots are greater by 2 than those of the equation $x^4 - 18x^3 - 32x^2 + 17x + 9 = 0.$

Ans. $y^4 - 26y^3 + 100y^2 - 103y + 7 = 0.$

4. Find an equation whose roots are greater by 1.2 than those of the equation $x^5 - 7x^3 + 2x - 8 = 0.$

Ans. $y^5 - 6y^4 + 7.4y^3 + 7.92y^2 - 17.872y - .79232 = 0.$

111. Transform each of the following equations into an equation wanting the second term:

$x^3 - 6x^2 + 7x - 2 = 0.$ *Ans.* $y^3 - 5y - 4 = 0.$

$x^3 - 6x^2 + 5 = 0.$ *Ans.* $y^3 - 12y - 11 = 0.$

3. $x^4 - 3x^3 + 5x - 6 = 0$.

Ans. $y^4 - \frac{27}{8}y^3 + \frac{13}{8}y - \frac{819}{256} = 0$.

4. $x^4 + 8x^3 + x^2 - x - 10 = 0$.

Ans. $y^4 - 23y^2 + 59y - 52 = 0$

THEOREM OF DESCARTES.

112.—1. Show that the equation $x^3 + 10x + 25 = 0$ has only one real root.

2. All the roots of the equation $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ are real; how many of them are negative?

Ans. None.

3. All the roots of the equation $x^5 - 6x^4 - 2x^3 + 2x^2 + 6x - 1 = 0$ are real; how many of them are positive?

Ans. Three.

4. All the roots of the equation $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$ are real; how many of them are positive? *Ans.* All are positive.

DERIVED FUNCTIONS.

113.—1. Find the derivatives of $x^5 - px^3 - qx^2 + s$.

Ans. $\begin{cases} 5x^4 - 3px^2 - 2qx, \\ 20x^3 - 6px - 2q, \\ 60x^2 - 6p, \\ 120x, \\ 120. \end{cases}$

2. Transform the equation $x^3 - 6x^2 + 7x - 2 = 0$ into another wanting the second term by the method of derivatives. *Ans.* $x^3 - 5x - 4 = 0$.

3. Find the first derivative of $(x+a)^2(x+b)^3$.

Ans. $3(x+a)^2(x+b)^2+2(x+b)^3(x+a)$.

4. Find the first derivative of $(x-a)^m(x-b)^n(x-c)^p$.

Ans. $m(x-a)^{m-1}(x-b)^n(x-c)^p+n(x-b)^{n-1}(x-a)^m(x-c)^p+p(x-c)^{p-1}(x-a)^m(x-b)^n$.

ROOTS COMMON TO TWO EQUATIONS.

114.—1. Find the root which is common to the two equations $x^3-3x^2-16x-12=0$ and $x^3-7x^2+5x+13=0$. *Ans.* -1.

2. Find the root which is common to the two equations $x^3-3x^2+11x-9=0$ and $x^3-5x^2+11x-7=0$. *Ans.* 1.

3. Find the roots common to the two equations $x^4-2x^3-7x^2+26x-20=0$ and $x^4+4x^3-2x^2-12x+8=0$. *Ans.* $-1\pm\sqrt{5}$.

4. The sum of two of the roots of the equation $x^4-2x^3-2x^2+8x-8=0$ is zero; solve the equation. *Ans.* $x=\pm 2$ or $1\pm\sqrt{-1}$.

EQUAL ROOTS.

115. Find the equal roots of each of the following equations:

1. $x^5+2x^4-11x^3-8x^2+20x+16=0$. *Ans.* 2, 2, -1, -1.
2. $x^5-2x^4+3x^3-7x^2+8x-3=0$. *Ans.* 1, 1, 1.
3. $x^3+x^2-16x+20=0$. *Ans.* 2, 2.
4. $x^4+2x^3-3x^2-4x+4=0$. *Ans.* 1, 1, -2, -2.
5. $x^3-5x^2+10x-8=0$. *Ans.* It has no equal roots.

LIMITS OF THE ROOTS OF AN EQUATION.

116. Find the first figure of one of the roots of each of the following equations:

1. $x^3 + 6x^2 + 27x - 26 = 0$. *Ans.* .8.
2. $x^3 - 9x^2 + 6x - 2 = 0$. *Ans.* 8.
3. $x^3 - 3x^2 + 5x - 43 = 0$. *Ans.* 4.
4. $x^4 - 5x^3 + 9x = 2.8$. *Ans.* .3.

117. Find the limits of the positive roots in each of the following equations:

1. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0$. *Ans.* 6 and $\frac{39}{44}$.
2. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0$. *Ans.* 8 and $\frac{1}{5}$.
3. $x^4 + 11x^2 - 25x - 67 = 0$.

Ans. $1 + \sqrt[3]{67}$ and $\dfrac{1}{1 + \sqrt{\frac{11}{67}}}$.

4. $3x^3 - 2x^2 - 11x + 4 = 0$. *Ans.* $\frac{14}{3}$ and $\frac{4}{15}$.

118. Find the limits of the negative roots in each of the following equations:

1. $x^4 + 5x^3 + 37x^2 + 3x + 39 = 0$.

Ans. -6 and $-\frac{39}{44}$.

2. $x^5 - 7x^4 - 12x^3 + 49x^2 + 52x + 13 = 0$.

Ans. -8 and $-\frac{1}{5}$.

3. $x^4 + 11x^3 + 25x - 67 = 0$.

Ans. $-(1 + \sqrt[3]{67})$ and $-\dfrac{1}{1 + \sqrt{\dfrac{11}{67}}}$.

4. $3x^3 + 2x^2 - 11x - 4 = 0$. *Ans.* $-\frac{14}{3}$ and $-\frac{4}{15}$.

STURM'S THEOREM.

119. Find the number and situation of the real roots of each of the following equations:

1. $x^3 + 2x^2 - 3x + 2 = 0$.

Ans. One; between -4 and -3.

2. $x^3 - 2x - 5 = 0$. *Ans.* One; between 2 and 3.

3. $x^3 - 3x^2 - 4x + 11 = 0$.

Ans. Three; one between -2 and -1, one between 1 and 2, and one between 3 and 4.

4. $x^4 - 4x^3 - 3x + 23 = 0$.

Ans. Two; one between 2 and 3, and one between 3 and 4.

5. $x^4 - 33x^2 - 100x - 84 = 0$.

Ans. $x = 7, -2, -2,$ or -3.

6. $x^4 + x^3 + x^2 - x - 500 = 0$.

Ans. Two; one between -5 and -4, and one between 4 and 5.

7. $x^4 - 12x^2 + 12x - 3 = 0$.

Ans. Two between 0 and 1, one between 2 and 3, and one between -4 and -3.

8. $x^4 - x^3 - 4x^2 + 4x + 1 = 0$.

HORNER'S METHOD OF APPROXIMATION.

120. Find a root situated between the assigned limits in each of the following equations:

1. $x^3 + 2x^2 - 23x = 70$; root between 5 and 6.

Ans. 5.1345.

2. $x^3 - 17x^2 + 42x = 185$; root between 15 and 16.

Ans. 15.024.

3. $3x^3 + 2x^2 + 4x = 75$; root between 2 and 3.

Ans. 2.5779.

4. $x^4 + x^3 + x^2 - x - 500 = 0$; root between 4 and 5.

Ans. 4.4604.

5. $x^4 - 9x^3 - 11x^2 - 20x = -4$; root between .1 and .2.

Ans. .1796.

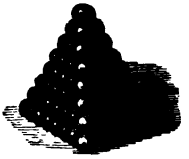
6. $x^4 - 3x^3 + 75x = 10000$; root between 9 and 10.

Ans. 9.8860.

APPENDIX.

PILES OF BALLS.

1. To find the number of balls in a triangular pile.



The number of balls in the first or top course is 1.

The number of balls in the 2d course is $1 + 2 = 3$.

The number of balls in the 3d course is $1 + 2 + 3 = 6$.

The number of balls in the 4th course is $1+2+3+4=10$.

. . . . . .

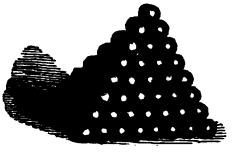
The number of balls in the n^{th} course is $1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$.

Hence, the number of balls in a triangular pile of n courses is equal to the sum of n terms of the series 1, 3, 6, 10, ...

Denoting this sum by s_n, we have

$$s_n = \frac{n(n+1)(n+2)}{\lfloor 3} \text{ (Alg., 533, 5).}$$

2. To find the number of balls in a square pile.



The number of balls in the 1st course is 1.

The number of balls in the 2d course is $2^2 = 4$.

The number of balls in the 3d course is $3^2 = 9$.

The number of balls in the 4th course is $4^2 = 16$.

. . . . . .

The number of balls in the n^{th} course is n^2.

Hence, the number of balls in a square pile of n courses is equal to the sum of n terms of the series 1, 4, 9, 16 . . . .

Denoting this sum by s_n, we have

$$s_n = \frac{n\,(n+1)\,(2n+1)}{\lfloor 3}.$$

3. To find the number of balls in a rectangular pile.

Denote the number of balls in the length of the base by m, and the number in the breadth of the base by n. Then the rectangular pile is made up of a square pile of n courses and $m - n$ triangular strata. The number of balls in the square pile is $\frac{n\,(n+1)\,(2n+1)}{\lfloor 3}$, and the number of balls

in each of the triangular strata is $\frac{n(n+1)}{2}$. Hence, denoting the number of balls in the rectangular pile by s_n, we have

$$s_n = \frac{n(n+1)(2n+1)}{\lfloor 3} + \frac{(m-n)n(n+1)}{2}$$

$$= \frac{n(n+1)(3m-n+1)}{\lfloor 3}.$$

4. ***EXAMPLES.***

1. Find the number of balls in a triangular pile of 15 courses.

Substituting 15 for n in the formula of Art. **1**, we have

$$s_{15} = \frac{15(15+1)(15+2)}{2 \times 3} = 680.$$

2. Find the number of balls in an incomplete triangular pile of 15 courses, having 21 balls in the upper course.

Denote the number of courses necessary to complete the pile by x; then $\frac{(x+1)(x+2)}{2} = 21$; whence $x = 5$. The completed pile would therefore contain 20 courses. The number of balls in a triangular pile of 20 courses is 1540, and the number of balls in a triangular pile of 5 courses is 35; hence the number in the incomplete triangular pile is 1505.

3. Find the number of balls in a square pile of 15 courses.

Ans. 1240.

4. Find the number of balls in a rectangular pile, the length of the base being 52 balls and the breadth 34 balls.

Ans. 24395.

5. Find the number of balls in an incomplete square pile of 9 courses, having 49 balls in the upper course.

Ans. 1149.

6. Find the number of balls in an incomplete square pile having 900 balls in the lower course and 225 balls in the upper course. *Ans.* 8440.

7. The number of balls in a rectangular pile of 20 courses is 6440; how many balls are in its base? *Ans.* 740.

8. The number of balls in a triangular pile is to the number in a square pile of the same number of courses as 6 is to 11; find the number of balls in each pile.

Ans. 816, and 1496.

DEVELOPMENT OF SIMPLE NUMERICAL SURDS.

5. The approximate value of a simple numerical surd may be found by means of the Binomial Formula.

Suppose the given surd to be of the n^{th} degree. Let a^n denote the perfect n^{th} power which is next less or next greater than the given number, and let b denote the difference between this power and the given number; then will $\sqrt[n]{a^n \pm b}$ denote the given surd.

Developing $\sqrt[n]{a^n \pm b}$, we have

$$\sqrt[n]{a^n \pm b} = a\left(1 \pm \frac{1}{n}\cdot\frac{b}{a^n} + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{b^2}{a^{2n}}\right.$$

$$\left. \pm \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}\cdot\frac{b^3}{a^{3n}} + \ldots\ldots\right) \quad \ldots\ldots \quad \text{(R)}.$$

Hence,

Any simple numerical surd may be developed into a series of rational terms.

EXAMPLES.

Find the value of each of the following surds to the 5th decimal place:

1. $\sqrt[3]{76}$.

In this example $n = 3$, and since $\sqrt[3]{76} = \sqrt[3]{4^3 + 12}$, we have $a = 4$ and $b = 12$; hence,

$$\frac{b}{a^n} = \frac{3}{16'} \quad \frac{1}{n} = \frac{1}{3'} \quad \frac{1-n}{2n} = -\frac{1}{3'} \quad \frac{1-2n}{3n} = -\frac{5}{9'}$$

$$\frac{1-3n}{4n} = -\frac{2}{3'} \quad \frac{1-4n}{5n} = -\frac{11}{15'} \quad \frac{1-5n}{6n} = -\frac{7}{9'}$$

$$\frac{1-6n}{7n} = -\frac{17}{21'} \ldots .$$

Denote the terms within the parenthesis in (R) by A, B, C, D, ...; then in this example

$$A = +1.0000000$$

$$B = +\frac{1}{3}\cdot\frac{3}{16} = +.0625000$$

$$C = -\frac{1}{3}\cdot\frac{3}{16}B = -.0039062$$

$$D = -\frac{5}{9}\cdot\frac{3}{16}C = +.0004069$$

$$E = -\frac{2}{3}\cdot\frac{3}{16}D = -.0000508$$

$$F = -\frac{11}{15}\cdot\frac{3}{16}E = +.0000069$$

$$G = -\frac{7}{9}\cdot\frac{3}{16}F = -.0000010$$

$$H = -\frac{17}{21}\cdot\frac{3}{16}G = +.0000001$$

. . . . . .

Whence, $A+B+C+D+E+F+G+H+\ldots = \overline{1.0589559}$;

$\therefore\ \sqrt[3]{76} = 4 \times 1.0589559 = 4.23582 +.$

2. $\sqrt[5]{25}$. *Ans.* 1.90365.

3. $\sqrt[3]{9}$. *Ans.* 2.08008.

4. $\sqrt[3]{31}$. *Ans.* 3.14138.

5. $\sqrt[3]{100}$. *Ans.* 4.64158.

COMPOUND INTEREST.

6. To find the amount of p dollars, at compound interest, for n years, at r per cent. per annum, when the interest is compounded every instant.

Denoting the amount by A and the number of instants in a year by m, we have

$$A = p\left(1 + \frac{r}{m}\right)^{mn} \quad . \quad . \quad . \quad (1) \text{ (Alg., 578)};$$

whence,

$$\log_e A = \log_e p + mn \log_e \left(1 + \frac{r}{m}\right)$$

$$= \log_e p + mn\left(\frac{r}{m} - \frac{r^2}{2m^2} + \frac{r^3}{3m^3} - \ldots\right) \text{ (Alg., 567)}$$

$$= \log_e p + n\left(r - \frac{r^2}{2m} + \frac{r^3}{3m^2} - \ldots\right) \quad . \quad . \quad . \quad (2).$$

But by hypothesis $m = \infty$; hence (2) becomes

$$\log_e A = \log_e p + nr \quad . \quad . \quad . \quad (3);$$

whence, $$\log_e A - \log_e p = nr \quad . \quad . \quad . \quad (4);$$

whence, $$\log_e \frac{A}{P} = nr \quad . \quad . \quad . \quad (5);$$

that is, $$\frac{A}{P} = e^{nr} \quad . \quad . \quad . \quad (6);$$

whence, $$A = pe^{nr} \quad . \quad . \quad . \quad (7).$$

ELIMINATION BY THE METHOD OF THE GREATEST COMMON DIVISOR.

7. One of the most general methods for the elimination of unknown quantities from a group of equations is that known as *The Method of the Greatest Common Divisor.*

Let it be required to eliminate x from the two equations

$$f(x, y) = 0 \quad \text{and} \quad f'(x, y) = 0.$$

Suppose that $f(x, y)$ and $f'(x, y)$ are arranged according to the descending powers of x, and that $f'(x, y)$ is not of a higher degree than $f(x, y)$ with reference to x.

We now apply to $f(x, y)$ and $f'(x, y)$ the process of finding their G. C. D., and continue the process until a remainder is obtained which is independent of x. The equation obtained by putting this remainder equal to zero will be the required equation. For, denoting the successive quotients by $q_1, q_2, q_3, \ldots . q_n$, and the corresponding remainders by $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y), \ldots . f_n(y)$, we have

$$f(x, y) = q_1 f'(x, y) + f_1(x, y) \quad . \ . \ . \quad (1)$$

$$f'(x, y) = q_2 f_1(x, y) + f_2(x, y) \quad . \ . \ . \quad (2)$$

$$f_1(x, y) = q_3 f_2(x, y) + f_3(x, y) \quad . \ . \ . \quad (3),$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$f_{n-2}(x, y) = q_n f_{n-1}(x, y) + f_n(y) \quad . \ . \ . \quad (n).$$

Now since $f(x, y) = 0$ and $f'(x, y) = 0$, it follows from (1) that $f_1(x, y) = 0$; hence by (2) $f_2(x, y) = 0$; hence by (3) $f_3(x, y) = 0$; and so on; hence $f_n(y) = 0$. If the number of given simultaneous equations is greater

than two, we may eliminate one of the unknown quantities by combining one of the given equations with each of the others, and thus obtain a new group containing one equation less than the given group. We may then apply the same process to the new group, and so on until a single equation containing only one unknown quantity is obtained.

EXAMPLES.

1. Eliminate x from the equations

$$x + y - 5 = 0 \quad . \quad . \quad . \quad (1),$$
$$x^2 + y^2 - 13 = 0 \quad . \quad . \quad . \quad (2).$$

$$\begin{array}{r|l} x^2 + y^2 - 13 & x + y - 5 \\ \hline x^2 + (y-5)x & x - (y-5) \\ -(y-5)\,x + y^2 - 13 & \\ -(y-5)\,x - y^2 + 10y - 25 & \\ \hline 2y^2 - 10y + 12 = 0 & \end{array}$$

2. Eliminate x from the equations

$$x^2 + xy + y^2 = 1,$$
$$x^3 + y^3 = 0.$$
$$Ans.\ 4y^6 - 6y^4 + 3y^2 - 1 = 0.$$

3. Eliminate x from the equations

$$x^3 + x^2y + x + y = 4,$$
$$x^3 + x^2 + xy = 3.$$
$$Ans.\ y - 1 = 0, \text{ or } y^2 - 3y + 21 = 0.$$

4. Eliminate x from the equations

$$x^2 + y^2 - (x + y) = 78,$$
$$x + y + xy = 39.$$
$$Ans.\ y^4 + y^3 - 77y^2 - 273y + 1404 = 0.$$

5. Eliminate x and y from the equations

$$x + y + z = a,$$
$$xz + xy + yz = b,$$
$$xyz = c.$$

Ans. $z^3 - az^2 + bz - c = 0.$

6. Eliminate x and y from the equations

$$x + y^2 = 4,$$
$$y + z^2 = 2,$$
$$z + x^2 = 10.$$

Ans. $z^8 - 8z^6 + 16z^4 + z - 10 = 0.$

INTEGRAL ROOTS.

8. Any equation having fractional coefficients can be transformed into another having all its coefficients integers, and the coefficient of its first term unity. The transformed equation cannot have a rational fraction for a root (Alg., **594**); hence, if it has a rational root, that root must be an integer. Again, the independent term of the equation is divisible by each of the roots (Alg., **593,** Cor. 5); hence, no integer which is not a factor of the independent term can be a root of the equation.

EXAMPLES.

1. Find the integral roots of the equation

$$x^3 - 3x^2 - 8x - 10 = 0.$$

Here 11 is a superior limit of the positive roots (Alg., **609**), and -3 is an inferior limit of the negative roots

(Alg., **611**); hence, all the real roots of the given equation lie between 11 and -3. The divisors of the independent term 10 which lie between these limits are 10, 5, 2, 1, -1, and -2.

If 10 is a root of the equation, its first member must be divisible by $x - 10$. When we attempt to divide $x^3 - 3x^2 - 8x - 10$ by $x - 10$, we find the remainder to be -790; thus,

$$\begin{array}{r} 1 - 3 - 8 - 10 \,\big|\, \underline{10} \\ +\ 10 - 70 - 780 \\ \hline 1 - 7 - 78 - 790; \end{array}$$

hence 10 is not a root.

The number 5 satisfies the condition of a root; thus,

$$\begin{array}{r} 1 - 3 - 8 - 10 \,\big|\, \underline{5} \\ +\ 5 + 10 + 10 \\ \hline 1 + 2 + 2 + 0 \end{array}$$

It will be found by trial that 2, 1, -1, and -2 are not roots; hence, the equation has only one integral root.

The two other roots can be found from the equation

$$x^2 + 2x + 2 = 0.$$

2. Find the integral roots of the equation

$$x^3 - 9x^2 + 26x - 24 = 0.$$

This equation has no negative roots (Alg., **602**). The limits of the positive roots are 25 and $\frac{12}{25}$ (Alg., **609-610**). The numbers 2, 3, and 4, are proved to be roots as follows:

$$\begin{array}{r|l} 1-9+26-24 & 2 \\ \quad +2-14+24 & \\ \hline 1-7+12+0 & 3 \\ \quad +3-12 & \\ \hline 1-4+0 & 4 \\ \quad +4 & \\ \hline 1+0 & \end{array}$$

3. Find the integral roots of the equation

$$x^5+5x^4+x^3-16x^2-20x-16=0.$$

Ans. 2, -2, -4.

4. Find the integral roots of the equation

$$x^3-6x^2+11x-6=0.$$

Ans. 1, 2, 3.

5. Find the integral roots of the equation

$$x^4+4x^3-x^2-16x-12=0.$$

Ans. 2, -1, -2, -3.

6. Find the integral roots of the equation

$$x^4-6x^3-16x+21=0.$$

Ans. 1, 3.

7. Find all the roots of the equation

$$x^4-6x^3+5x^2+2x-10=0.$$

Ans. -1, 5, $1\pm\sqrt{-1}$.

8. Find all the roots of the equation

$$x^3-106x-420=0.$$

9. Find all the roots of the equation

$$x^3-2x^2-25x+50=0.$$

10. Find all the roots of the equation

$$x^4+12x^3+47x^2+72x+36=0.$$

Ans. -1, -2, -3, -6.

11. Eliminate x from the equations

$$x^2+y = 7,$$
$$x +y^2=11,$$

and find an integral root of the resulting equation.

Ans. $y = 3$.

In the following examples the integral roots consist of two or more figures.

12. Find all the roots of the equation

$$x^3 - 237x^2 + 4x - 948 = 0.$$

This equation has a real root lying between 200 and 300; hence, 2 is the first figure of that root. The other figures are found as follows:

$$\begin{array}{rrrr|l}
1 & -237 & +4 & -948 & 200 + 30 + 7 = 237 \\
 & +200 & -7400 & -1479200 & \\
\hline
 & -37 & -7396 & -1480148^{(1)} & \\
 & +200 & +32600 & & \\
\hline
 & +163 & +25204^{(1)} & & \\
 & +200 & & & \\
\hline
 & +363^{(1)} & & & \\
\hline
1^{(1)} & +363^{(1)} & +25204^{(1)} & -1480148^{(1)} & \\
 & +30 & +11790 & +1109820 & \\
\hline
 & +393 & +36994 & -370328^{(2)} & \\
 & +30 & +12690 & & \\
\hline
 & +423 & +49684^{(2)} & & \\
 & +30 & & & \\
\hline
 & 453^{(2)} & & & \\
\hline
1^{(2)} & +453^{(2)} & +49684^{(2)} & -370328^{(2)} & \\
 & +7 & +3220 & +370328 & \\
\hline
 & 460 & +52904 & +0 &
\end{array}$$

The numbers marked (1) are the coefficients of an equation whose roots are less by 200 than those of the given equation (Alg., **598**); hence, the first transformed equation is $y^3 + 363y^2 + 25204y - 1480148 = 0$. This equation has one root between 30 and 40; hence, 3 is the second figure of the required root. The numbers marked (2) are the coefficients of an equation whose roots are less by 30 than those of the first transformed equation; hence, the second transformed equation is $z^3 + 453z^2 + 49684z - 370328 = 0$. By trial we find that 7 is a root of this equation; hence, 237 is a root of the given equation. The other roots may now be found by dividing the given equation by $x - 237$ and solving the resulting equation.

13. Find all the roots of the equation

$$x^3 + 125x^2 + 4x + 500 = 0.$$

The integral root of this equation is negative. Change the signs of the alternate terms and proceed with the result as in the 12th example.

14. Find the integral root of the equation

$$x^5 - 15x^4 + 13x^3 - 195x^2 + 36x - 540 = 0.$$

15. Find the integral root of the equation

$$x^4 - 20x^3 - 67x^2 - 41x - 115 = 0.$$

16. Find the integral roots of the equation

$$x^4 + 100x^3 - 3121x^2 + 400x - 12500 = 0.$$

CUBIC EQUATIONS.

9. Every cubic equation containing only one unknown quantity can be reduced to the form of

$$x^3 + ax = b \quad . \quad . \quad . \quad (1) \text{ (Alg., } \mathbf{599}\text{).}$$

Assuming $x = \sqrt[3]{y} - \dfrac{a}{3\sqrt[3]{y}}$, (1) becomes

$$y^2 - by = \frac{a^3}{27} \quad . \quad . \quad . \quad (2);$$

whence, $$y = \frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \quad . \quad . \quad . \quad (3);$$

$$\therefore \quad x = \left(\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{\frac{1}{3}} - \frac{a}{3\left(\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{\frac{1}{3}}}$$

$$= \left(\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{\frac{1}{3}} + \left(\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{\frac{1}{3}} \cdots (4).$$

This is Cardan's formula.

If a is negative and $\dfrac{a^3}{27}$ is numerically greater than $\dfrac{b^2}{4}$, Cardan's formula fails.

EXAMPLES.

Solve the following equations:

1. $x^3 - 9x - 14 = 0$. *Ans.* $x = 3.591$.
2. $x^3 - 9x + 28 = 0$. *Ans.* $x = -4$ or $2 \pm \sqrt{-3}$.
3. $x^3 + 6x - 2 = 0$. *Ans.* $x = \sqrt[3]{4} - \sqrt[3]{2}$.
4. $x^3 - 3x - 18 = 0$. *Ans.* $x = 3$ or $\frac{1}{2}(-3 \pm \sqrt{-15})$.
5. $x^3 - 9x^2 + 25x - 25 = 0$. *Ans.* $x = 5$ or $2 \pm \sqrt{-1}$.
6. $x^3 + 3x^2 + 9x - 13 = 0$. *Ans.* $x = 1$ or $-2 \pm 3\sqrt{-1}$.

RECIPROCAL EQUATIONS.

10. ***A Reciprocal Equation*** is one which is not changed when the unknown quantity is changed into its reciprocal. Hence, if a is a root of such an equation, $\frac{1}{a}$ is also a root.

A reciprocal equation is sometimes called a *Recurring Equation.*

11. To find the conditions which a proposed equation must satisfy in order that it may be a reciprocal equation.

Let the proposed equation be

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + Jx^2 + Kx + L = 0 \ldots (1).$$

Substituting $\frac{1}{x}$ for x, and multiplying the result by $\frac{x^n}{L}$, we obtain

$$x^n + \frac{K}{L}x^{n-1} + \frac{J}{L}x^{n-2} + \ldots + \frac{B}{L}x^2 + \frac{A}{L}x + \frac{1}{L} = 0 \ldots (2).$$

Now, in order that (1) may be identical with (2), we must have

$$A = \frac{K}{L}, \quad B = \frac{J}{L}, \ldots J = \frac{B}{L}, \quad K = \frac{A}{L}, \quad L = \frac{1}{L} \ldots (3).$$

From the equation $L = \frac{1}{L}$ we obtain $L = \pm 1$. These values of L give rise to two classes of reciprocal equations.

I. Suppose $L = 1$; then equations (3) become

$$A = K, \quad B = J, \ldots J = B, \quad K = A, \quad L = 1;$$

hence, an equation is a reciprocal equation if the coefficients

of the terms equidistant from the first and last are equal. Thus, $3x^3 + 5x^2 + 5x + 3 = 0$ is a reciprocal equation of the first class.

II. Suppose $L = -1$; then equations (3) become

$$A = -K,\ B = -J,\ \ldots J = -B,\ K = -A,\ L = -1.$$

In this case, if the equation is of an even degree, we have among the conditions the equation $D = -D$, where D represents the coefficient of the middle term; but this condition is impossible unless $D = 0$; hence, an equation is a reciprocal equation if the coefficients of the terms equidistant from the first and last are numerically equal and have contrary signs, with the condition that the coefficient of the middle term shall be zero if the equation is of an even degree.

Cor. 1.—The last term of every reciprocal equation of the first class is positive, and the last term of every reciprocal equation of the second class is negative.

Cor. 2.—One root of every reciprocal equation of the first class and of an *odd* degree is -1, as is obvious by inspection.

Cor. 3.—One root of every reciprocal equation of the second class and of an *odd* degree is $+1$.

Cor. 4.—Two roots of every reciprocal equation of the second class and of an *even* degree are $+1$ and -1.

12. To transform a reciprocal equation of an odd degree into one of an even degree of the first class.

1. Let $f(x) = 0$ be a reciprocal equation of the first class and of the n^{th} degree, and suppose n to be odd.

Now $f(x)$ must be such that

$$f(x) = \frac{x^n}{L} f\left(\frac{1}{x}\right) \quad . \quad . \quad . \quad (1) \text{ (App. 11).}$$

Dividing (1) by $x + 1$ (App. 11, Cor. 2), we obtain

$$\frac{f(x)}{x+1} = \frac{x^n}{L(x+1)} f\left(\frac{1}{x}\right) = \frac{x^{n-1}}{L} \cdot \frac{f\left(\frac{1}{x}\right)}{1+\frac{1}{x}} \quad . \quad . \quad . \quad (2).$$

But $\frac{x^{n-1}}{L} \cdot \frac{f\left(\frac{1}{x}\right)}{1+\frac{1}{x}}$ may be obtained from $\frac{f(x)}{x+1}$ by changing x into $\frac{1}{x}$ and multiplying the result by $\frac{x^{n-1}}{L}$; hence, $\frac{f(x)}{x+1} = 0$ is a reciprocal equation. Again, since $f(x)$ is of an odd degree, $\frac{f(x)}{x+1}$ is of an even degree; and since the last term of $f(x)$ is positive, the last term of the quotient $\frac{f(x)}{x+1}$ is positive. Therefore, if a reciprocal equation of an odd degree and of the first class be divided by $x + 1$, the result will be a reciprocal equation of an even degree of the first class.

2. Let $f(x) = 0$ be a reciprocal equation of the second class and of the n^{th} degree, and suppose n to be odd;

then $$f(x) = \frac{x^n}{L} f\left(\frac{1}{x}\right).$$

Dividing by $x-1$ (App. **11**, Cor. 3),

$$\frac{f(x)}{x-1}=\frac{x^n}{\mathrm{L}(x-1)}f\left(\frac{1}{x}\right)=\frac{x^{n-1}}{\mathrm{L}}\cdot\frac{f\left(\frac{1}{x}\right)}{1-\frac{1}{x}}=-\frac{x^{n-1}}{\mathrm{L}}\cdot\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x}-1}.$$

But $-\frac{x^{n-1}}{\mathrm{L}}\cdot\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x}-1}$ may be obtained from $\frac{f(x)}{x-1}$ by changing x into $\frac{1}{x}$, and multiplying the result by $-\frac{x^{n-1}}{\mathrm{L}}$; hence, $\frac{f(x)}{x-1}=0$ is a reciprocal equation of an even degree; and since the last term of $f(x)$ is negative, the last term of the quotient $\frac{f(x)}{x-1}$ must be positive. Therefore, if a reciprocal equation of an odd degree and of the second class be divided by $x-1$, the result will be a reciprocal equation of an even degree of the first class.

13. To transform a reciprocal equation of an even degree and of the second class into one of an even degree of the first class.

Let $f(x)=0$ be a reciprocal equation of the second class and of the n^{th} degree, and suppose n to be even;

then $$f(x)=\frac{x^n}{\mathrm{L}}f\left(\frac{1}{x}\right).$$

Dividing by x^2-1 (App. **11**, Cor. 4),

$$\frac{f(x)}{x^2-1}=\frac{x^n}{\mathrm{L}(x^2-1)}f\left(\frac{1}{x}\right)=-\frac{x^{n-2}}{\mathrm{L}}\cdot\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x^2}-1}.$$

But $-\frac{x^{n-2}}{L}\cdot\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x^2}-1}$ may be obtained from $\frac{f(x)}{x^2-1}$ by changing x into $\frac{1}{x}$, and multiplying the result by $-\frac{x^{n-2}}{L}$; hence, $\frac{f(x)}{x^2-1}=0$ is a reciprocal equation of an even degree; and since the last term of $f(x)$ is negative, the last term of the quotient $\frac{f(x)}{x^2-1}$ will be positive.

14. To solve a reciprocal equation.

It follows from Arts. **12** and **13** that a reciprocal equation of an even degree and of the first class may be considered as the standard form of reciprocal equations.

Let $$x^{2n}+Ax^{2n-1}+Bx^{2n-2}+Cx^{2n-3}+\ldots$$
$$+Cx^3+Bx^2+Ax+1=0 \quad . \quad . \quad . \quad (1)$$

be the proposed equation.

Dividing (1) by x^n and collecting, in pairs, the terms of the result which are equidistant from the extremes, we have

$$x^n+\frac{1}{x^n}+A\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+B\left(x^{n-2}+\frac{1}{x^{n-2}}\right)$$
$$+C\left(x^{n-3}+\frac{1}{x^{n-3}}\right)+\ldots=0 \quad . \quad . \quad . \quad (2).$$

Assuming $x+\frac{1}{x}=y$, then

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)-2=y^2-2,$$

$$x^3+\frac{1}{x^3}=\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)-\left(x+\frac{1}{x}\right)$$
$$=(y^2-2)\,y-y=y^3-3y,$$

$$x^4+\frac{1}{x^4}=\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)-\left(x^2+\frac{1}{x^2}\right)$$
$$=(y^3-3y)y-(y^2-2)=y^4-4y^2+2,$$

. . . . . .

$$x^n+\frac{1}{x^n}=\left(x^{n-1}+\frac{1}{x^{n-1}}\right)\left(x+\frac{1}{x}\right)-\left(x^{n-2}+\frac{1}{x^{n-2}}\right).$$

Substituting y for $x+\frac{1}{x}$, y^2-2 for $x^2+\frac{1}{x^2}$, y^3-3y for $x^3+\frac{1}{x^3}$, ... we obtain an equation of the n^{th} degree. If this equation can be solved, the values of x may be found from the equation $x+\frac{1}{x}=y$.

EXAMPLES.

1. Solve the equation

$$2x^6+x^5-13x^4+13x^3-x-2=0 \quad . \quad . \quad . \quad (1).$$

This equation is of an even degree and of the second class; hence, two of its roots are $+1$ and -1.

Dividing (1) by x^2-1,

$$2x^4+x^3-11x^2+x+2=0 \quad . \quad . \quad . \quad (2).$$

Dividing (2) by $2x^2$ and collecting terms,

$$x^2+\frac{1}{x^2}+\frac{1}{2}\left(x+\frac{1}{x}\right)-\frac{11}{2}=0 \quad . \quad . \quad . \quad (3).$$

Assume $x + \frac{1}{x} = y$; then

$$y^2 - 2 + \frac{y}{2} - \frac{11}{2} = 0 \quad . \quad . \quad . \quad (4);$$

whence, $y = \frac{5}{2}$ or -3.

Hence, $x + \frac{1}{x} = \frac{5}{2}$, or $x + \frac{1}{x} = -3$;

whence, $x = 2$ or $\frac{1}{2}$, or $\frac{1}{2}(-3 \pm \sqrt{5})$.

2. Solve the equation

$$x^6 + 3x^5 - 7x^4 + 6x^3 - 7x^2 + 3x + 1 = 0.$$

Ans. $x = \pm\sqrt{-1}$, 1, $\frac{1}{2}(-5 \pm \sqrt{21})$.

3. Solve the equation

$$2x^4 - 3x^3 - x^2 - 3x + 2 = 0.$$

Ans. $x = 2$, $\frac{1}{2}$, $\frac{1}{2}(-1 \pm \sqrt{-3})$.

4. Solve the equation $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$.

5. Solve the equation $x^4 + 4x^3 - 5x^2 + 4x + 1 = 0$.

6. Solve the equation $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0$.

7. Solve the equation $x^4 + 4x^3 - 10x^2 + 4x + 1 = 0$.

8. Solve the equation $x^5 - 2x^4 - 19x^3 - 19x^2 - 2x + 1 = 0$.

9. Solve the equation $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0$.

10. Solve the equation $6x^5 - 11x^4 - 33x^3 + 33x^2 + 11x - 6 = 0$.

CONTINUED FRACTIONS.

15. ***A Continued Fraction*** is an expression of the form of

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}+\text{etc.},}}}$$

in which $a, b, c, d, \ldots$ are positive integers.

16. ***A Terminating Continued Fraction*** is one in which the number of simple fractions $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \ldots$ is *finite*.

17. ***An Infinite Continued Fraction*** is one in which the number of simple fractions is *infinite*.

18. ***A Converging Fraction or Convergent*** is the result obtained by considering a certain number of the simple fractions, and neglecting all that follow. Thus, in the continued fraction given in Art. **15** the first convergent is $\frac{1}{a}$; the second convergent is formed from $\cfrac{1}{a+\cfrac{1}{b}}$; it is therefore $\frac{b}{ab+1}$; the third convergent is formed from $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$; it is therefore $\frac{bc+1}{(ab+1)c+a}$; and so on.

A converging fraction is sometimes called *An Approximating Fraction.*

19. *The convergents of an odd order are greater, and those of an even order are less, than the value of the continued fraction.*

The first convergent $\frac{1}{a}$ is too large, because the denominator a is too small; the second convergent $\frac{b}{ab-1}$ is too small, because $a+\frac{1}{b}$ is too large; the third convergent $\frac{bc+1}{(ab+1)c+a}$ is too large, because $a+\cfrac{1}{b+\frac{1}{c}}$ is too small; and so on.

20. To find the value of a terminating continued fraction.

EXAMPLES.

1. Find the value of $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\frac{1}{d}}}}$.

$$c+\frac{1}{d}=\frac{cd+1}{d};\quad \therefore\ \frac{1}{c+\frac{1}{d}}=\frac{d}{cd+1}.$$

$$b+\frac{1}{c+\frac{1}{d}}=b+\frac{d}{cd+1}=\frac{b(cd+1)+d}{cd+1};$$

$$\therefore\ \frac{1}{b+\cfrac{1}{c+\frac{1}{d}}}=\frac{cd+1}{b(cd+1)+d}.$$

$$a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}=a+\frac{cd+1}{b(cd+1)+d}=$$

$$\frac{ad(bc+1)+ab+cd+1}{(bc+1)d+b};$$

$$\therefore\quad \frac{1}{a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}}=\frac{(bc+1)d+b}{ad(bc+1)+ab+cd+1}=$$

$$\frac{(bc+1)a+b}{(ab+1)cd+a(d+b)+1}.$$

2. Find the value of $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4}}}$ *Ans.* $\frac{13}{30}$.

3. Find the value of $\dfrac{1}{4+\dfrac{1}{3+\dfrac{1}{2}}}$ *Ans.* $\frac{7}{30}$.

4. Find the value of $1+\dfrac{1}{3+\dfrac{1}{5+\dfrac{1}{7+\dfrac{1}{9}}}}$ *Ans.* $\frac{1380}{1051}$.

21. To find approximate values of an infinite continued fraction.

EXAMPLES.

1. Find approximate values of

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}+\text{etc.}}}}$$

$\frac{1}{a}$ $= \frac{1}{a}$ the first convergent;

$\cfrac{1}{a+\cfrac{1}{b}}$ $= \frac{b}{ab+1}$ the second convergent;

$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$ $= \frac{bc+1}{(ab+1)c+a}$ the third convergent;

and so on.

Each of these convergents is nearer the true value of the continued fraction than the preceding one; for the second is what the first becomes by substituting for a the more accurate denominator $a+\frac{1}{b}$; the third is what the second becomes by substituting for b the more accurate denominator $b+\frac{1}{c}$; and so on.

2. Find the third convergent of

$$\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4}+\text{etc.}}}}$$

Ans. $\frac{19}{20}$.

3. Find the fourth convergent of

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}+\text{etc.}}}}$$

Ans. $\frac{8}{11}$.

4. Find the fourth convergent of

$$\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1}+\text{etc.}}}}}}$$

Ans. $\frac{6}{25}$.

22. To reduce a quantity to an expression containing a continued fraction.

Let Q represent the quantity to be reduced. Denote the greatest integer contained in Q by A, and the excess of Q above A by $\frac{1}{x}$; then

$$Q = A + \frac{1}{x}; \quad \text{whence,} \quad x = \frac{1}{Q - A}.$$

Denote the greatest integer contained in x by a, and the excess of x above a by $\frac{1}{x'}$; then

$$x = a + \frac{1}{x'}; \quad \text{whence,} \quad x' = \frac{1}{x - a}.$$

Denote the greatest integer contained in x' by a', and the excess of x' above a' by $\frac{1}{x''}$; then

$$x' = a' + \frac{1}{x''}; \quad \text{whence,} \quad x'' = \frac{1}{x' - a};$$

and so on. Hence,

$$Q = A + \cfrac{1}{a + \cfrac{1}{a' + \cfrac{1}{a''} + \text{etc.}}}$$

EXAMPLES.

1. Reduce $\frac{1380}{1051}$ to an expression containing a continued fraction.

In this example $A = 1$;

$$\therefore \quad x = \frac{1}{\frac{1380}{1051} - 1} = \frac{1}{\frac{329}{1051}} = \frac{1051}{329}; \quad \text{therefore} \quad a = 3.$$

$$\therefore \quad x' = \frac{1}{\frac{1051}{329} - 3} = \frac{1}{\frac{64}{329}} = \frac{329}{64}; \quad \text{therefore} \quad a' = 5.$$

$$\therefore \quad x'' = \frac{1}{\frac{329}{64} - 5} = \frac{1}{\frac{9}{64}} = \frac{64}{9}; \quad \text{therefore} \quad a'' = 7.$$

$$\therefore \quad x''' = \frac{1}{\frac{64}{9} - 7} = \frac{1}{\frac{1}{9}} = 9; \quad \text{therefore} \quad a''' = 9.$$

$$\therefore \quad \frac{1380}{1051} = 1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \frac{1}{9}}}}.$$

When the given quantity is a common fraction, A, a, a', a'', etc., are evidently the quotients which would be obtained by the process of finding the G. C. D. of the numerator and denominator. The operation may therefore be abridged as follows:

$$
\begin{array}{r}
1051)1380(1 = \mathrm{A} \\
\underline{1051} \\
329)1051(3 = a \\
\underline{987} \\
64)329(5 = a' \\
\underline{320} \\
9)64(7 = a'' \\
\underline{63} \\
1)9(9 = a''' \\
\underline{9}
\end{array}
$$

2. Reduce $\sqrt{6}$ to an expression containing a continued fraction.

In this example $\mathrm{A} = 2$;

$\therefore \quad x = \dfrac{1}{\sqrt{6}-2} = \dfrac{\sqrt{6}+2}{2}$; therefore $a = 2$.

$\therefore \quad x' = \dfrac{1}{\frac{\sqrt{6}+2}{2} - 2} = \dfrac{2}{\sqrt{6}-2} = \dfrac{2\sqrt{6}+4}{2} = \sqrt{6}+2$;

therefore $a' = 4$.

$\therefore \quad x'' = \dfrac{1}{\sqrt{6}+2-4} = \dfrac{1}{\sqrt{6}-2} = \dfrac{\sqrt{6}+2}{2}$;

therefore $a'' = 2$.

$\therefore \quad x''' = \dfrac{1}{\frac{\sqrt{6}+2}{2} - 2} = \sqrt{6}+2$; therefore $a''' = 4$; and so on.

$$\therefore\ \sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4}}}} + \text{etc.}$$

3. Reduce $\frac{445}{612}$ to a continued fraction.

4. Reduce $\sqrt{a^2+1}$ to an expression containing a continued fraction.

$$\textit{Ans. } a + \cfrac{1}{2a + \cfrac{1}{2a + \cfrac{1}{2a}}} + \text{etc.}$$

5. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

$$\textit{Ans. } \frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{39}{161},\ \ldots\ .$$

6. The ratio of the circumference of a circle to its diameter is 3.1415926535 +. Find approximate values for this ratio.

$$\textit{Ans. } 3,\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \ldots\ .$$

7. In 87969 years Mercury makes 277287 synodical revolutions. Find approximate values for the ratio of 87969 to 277287.

$$\textit{Ans. } \frac{1}{3},\ \frac{6}{19},\ \frac{7}{22},\ \frac{13}{41},\ \frac{33}{104}.$$

8. In 57551 years Venus makes 36000 synodical revolutions. Find approximate values for the ratio of 57551 to 36000.

$$\textit{Ans. } \frac{8}{5},\ \frac{235}{147}.$$

SUMMATION OF SERIES.

23. Denote the sum of a series whose r^{th} term is $\frac{q}{n(n+p)}$ by s, the sum of a series whose r^{th} term is $\frac{q}{n}$ by s', and the sum of a series whose r^{th} term is $\frac{q}{n+p}$ by s''; then, since

$$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right),$$

$$s = \frac{1}{p}(s' - s'').$$

EXAMPLES.

1. Find the sum of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots.$$

$$q = 1, \quad p = 1, \quad n = 1, 2, 3, 4, \ldots;$$

$$\therefore \quad s = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots.$$

$$= \left\{\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots. \\ \quad -\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.\right) \end{array}\right\} = 1$$

2. Find the sum of the first n terms of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots.$$

$$s = \left\{\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots. + \frac{1}{n} \\ \quad -\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots. + \frac{1}{n} + \frac{1}{n+1}\right) \end{array}\right\}$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

3. Find the sum of the series

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots. \qquad \textit{Ans.}\ \frac{1}{2}.$$

24. Denote the sum of a series whose r^{th} term is $\frac{q}{n(n+p)(n+2p)}$ by s, the sum of a series whose r^{th} term is $\frac{q}{n(n+p)}$ by s', and the sum of a series whose r^{th} term is $\frac{q}{(n+p)(n+2p)}$ by s''; then, since

$$\frac{q}{n(n+p)(n+2p)}=\frac{1}{2p}\left\{\frac{q}{n(n+p)}-\frac{q}{(n+p)(n+2p)}\right\},$$

$$s=\frac{1}{2p}(s'-s'').$$

EXAMPLES.

1. Find the sum of the series

$$\frac{4}{1\cdot 2\cdot 3}+\frac{5}{2\cdot 3\cdot 4}+\frac{6}{3\cdot 4\cdot 5}+\ldots.$$

$q=4,\ 5,\ 6,\ \ldots.,\qquad p=1,\qquad n=1,\ 2,\ 3,\ 4,\ \ldots.;$

$$\therefore\quad s=\frac{1}{2}\left\{\begin{array}{l}\frac{4}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{6}{3\cdot 4}+\ldots. \\ \qquad -\left(\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\ldots.\right)\end{array}\right\}$$

$$=\frac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots.\right)$$

$$=\frac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{1}{2}\right)=1\tfrac{1}{4}\ \text{(App., 23, 1)}.$$

2. Find the sum of the series

$$\frac{3}{5\cdot 8\cdot 11}+\frac{9}{8\cdot 11\cdot 14}+\frac{15}{11\cdot 14\cdot 17}+\ldots.$$

$$\textit{Ans.}\ \frac{13}{240}.$$

$x \div \sqrt[3]{\ }$

$\sqrt[3]{x^3} \div \sqrt[3]{x}$